EFFICIENT FITTING OF MULTI-PLANET KEPLERIAN MODELS TO RADIAL VELOCITY AND ASTROMETRY DATA

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ABSTRACT

We describe a technique for solving for the orbital elements of multiple planets from radial velocity (RV) and/or astrometric data taken with 1 m/s and μas precision, appropriate for efforts to detect Earth-massed planets in their stars’ habitable zones, such as NASA’s proposed Space Interferometry Mission. We include details of calculating analytic derivatives for use in the Levenberg-Marquardt (LM) algorithm for the problems of fitting RV and astrometric data separately and jointly.

We also explicate the general method of separating the linear and nonlinear components of a model fit in the context of an LM fit, show how explicit derivatives can be calculated in such a model, and demonstrate the speed up and convergence improvements of such a scheme in the case of a five-planet fit to published radial velocity data for 55 Cnc.

Subject headings: astrometry — planetary systems — methods: data analysis — methods: numerical — techniques: radial velocities

1. INTRODUCTION

1.1. Fitting Keplerian Curves

The discovery of over 27 multiple-planet systems in recent years (Wright et al. 2009) has required algorithms for disentangling the radial-velocity signature of such complex systems. Because the parameters describing a radial velocity (RV) or astrometric curve are nonlinear, there is no way to fit them analytically, and they must be found through an algorithmic search. Fitting for the Keplerian parameters of a single orbital companion is usually straightforward given a good period guess, for the Keplerian parameters of a single orbital companion must be found through an algorithmic search. Fitting for disentangling the radial-velocity signature of such complex systems. Because the parameters describing a radial velocity (RV) or astrometric signature of the system. In such cases where interactions are important, a full dynamical (Newtonian)

fit involving an n-body code must be used to properly fit the data and to ensure the short- and long-term stability of the solution. Even in these cases, a multi-planet Keplerian (kinematic) fit, which simply adds the reflex signatures of single planets and ignores planet-planet interactions, is still useful for efficiently identifying planets and providing good initial guesses to the n-body codes.

The Levenberg-Marquardt method (LM; Levenberg 1944; Marquardt 1963) is an efficient algorithm for finding a local minimum in a nonlinear χ² space (given good guesses), and is well-suited for the application of RV and astrometric fitting (Press et al. 1992). As a concrete example, we refer in this work to a useful IDL² implementation of this technique, MPFIT by Craig Markwardt³ (Markwardt 2009). Like most implementations of the algorithm, MPFIT requires a user-defined function that accepts, as an argument, trial values for the parameters being solved for, evaluates the model at those values, and returns the corresponding residuals to the data. MPFIT then uses this information to step through parameter space and locate the minimum in χ² using a combination of Newton’s method and a steepest-descent method. Uncertainties in these parameters can then be calculated by mapping the χ² space near this minimum (e.g. Wright et al. 2007) or through error “bootstrapping” (e.g. Butler et al. 2006).

The user-defined function in MPFIT also optionally returns the values of the derivatives of these residuals with respect to the parameters being fit, computed “analytically” or “explicitly”. Absent these derivatives, MPFIT will calculate numerical derivatives with small

¹ Townes Fellow, Space Sciences Laboratory, UC Berkeley
² IDL is a commercial programming language and environment by ITT Visual Information Solutions. http://www.ittvis.com/idl/
³ Available at http://purl.com/net/mpfit. MPFIT is a port of MINPACK-1 from FORTRAN, and is also available in C and Python.
steps in the fitted parameters and calculating the resulting change in the residuals. The user must choose step sizes that are not too large — thereby missing fine structure in the $\chi^2$ space — or too small — increasing compute time and potentially losing numerical precision. Adding explicit-derivative capability to MPFIT, or any implementation of the LM algorithm, obviates the need for explicitly providing step-sizes, and can greatly improve performance in terms of the number of steps taken and the CPU time consumed per step.

The LM algorithm is useful for finding the best-fitting parameters of a nonlinear model to a set of data from a “frequentist” perspective. A Bayesian approach can provide for more robust estimates of parameter uncertainties, especially when those uncertainties are large. One Bayesian method of exploring complex or high-dimensional spaces is the Markov chain Monte Carlo method (MCMC; Metropolis et al. 1953; Hastings 1970), which has been productively employed in the context of orbital fitting (e.g. Ford 2004; Driscoll & Fischer 2005; Balan & Lahav 2008).

Future space missions, such as NASA’s Space Interferometry Mission (SIM Lite), will obtain $\mu$as astrometry of nearby stars, referenced to an inertial astrometric grid. These measurements will be sufficiently precise to detect Earth-mass planets with orbital periods shorter than the mission lifetime. Since multiple-planet systems are common, interpreting these data in conjunction with precise radial velocity data will require algorithms that can efficiently and robustly search the large nonlinear parameter space of multiple-planet systems.

In this paper, we describe the method of efficiently fitting multi-Keplerian models to such high precision RV and astrometric data separating the parameters into linear and nonlinear components in the context of a LM algorithm, and provide the explicit derivatives used in such a fit. With modifications, the principles here can also be applied to MCMC methods, as well (Catanzarite, Zhai, & Shao 2009; Bakos et al. 2009, e.g.).

1.2. Plan

We begin with an elementary exposition to familiarize the reader with our notation (detailed in Table 2) and provide context for the later discussion. In §2, we explicate the method of exploiting linear parameters in the Kepler problem using the example of an RV time series. The calculation of explicit derivatives in §2.4 is general to any application of the LM algorithm where the model contains both linear and nonlinear parameters. We apply this method to the problem of astrometric data in §3 and specifically to the problem of combined astrometric and RV data in §3.3. We discuss nonlinear terms relevant to $\mu$as astrometric work and how to accommodate them in §4. In §5 we quantify the improvement in speed and convergence from exploiting linear parameters and from the use of explicit derivatives in the LM algorithm.

2. AN EXAMPLE: RADIAL VELOCITIES

2.1. Costs and Benefits of Exploiting Linear Parameters

When fitting RV data, there are $5n + 1$ Keplerian parameters to be fit, where $n$ is the number of planets: $P_j$, the period of the planet’s orbit; $K_j$, the semi-amplitude of the radial velocity signal; $e_j$, the eccentricity of the orbit; $\omega_j$, the argument of periastron; $\gamma_j$, a date of periastron passage; and $\gamma$, the apparent radial velocity of the center of mass of the system.

Exploiting linear parameters, as described below, reduces the search space to $3n$ dimensions (corresponding to $P$, $e$, and $t_p$ for each planet) and combines the other orbital elements into a set of linear parameters, which can be solved analytically (and therefore quickly and exactly) at each step in the search. The exact, analytic solution of these linear parameters greatly increases the speed and stability of search algorithms, but at a cost: the nonlinear parameters cannot be varied independently of the linear parameters.

Further reduction in the number of nonlinear parameters per planet is certainly possible. By exploiting an epicyclic or harmonic series expansion one can reduce the problem to only one nonlinear parameter per planet, $P_j$, Cumming et al. (2003) and Ford (2008) analyze the RV problem in the case of a circular orbits, and discuss the relationship between periodograms and Bayesian approaches to orbit fitting. Konacki & Maciejewski (1999) pursued a method analogous to ours in their approach to RV curve fitting, and Konacki, Maciejewski, & Wolszczan (2002) did the same for astrometry. Such approximations offer a different set of costs and benefits to the one presented here. For instance, when $K$ and $e$ are both large, it may require a large number of terms to adequately describe a set of RV data. We may pursue such an approach in a future version of our code.

The linear parameters in our treatment are not coefficients in a series expansion; rather, we recast the problem, separating linear parameters, which can be solved for exactly with linear algebra, from nonlinear parameters, which must be solved for algorithmically (using LM). As described in §2.2, the linear parameters are algebraic combinations of $K_j$, $\omega_j$, $\gamma_j$, and an optional trend parameter, while $P_j$, $t_p$, and $e$ are nonlinear parameters.

Two complications are introduced by exploiting linear parameters. First, because the linear parameters are computed analytically, and not algorithmically, their errors and covariances with the nonlinear parameters will not be computed naturally. (They also cannot be held fixed in the fit, although the issue of fixing the trend parameter can be finessed – see §2.6). The second complication is that computation of the explicit derivatives for the LM algorithm is not straightforward: in the context of the algorithm, the Keplerian model is not a simple function of the 3n nonlinear parameters being fit, but also depends on the data. For example, increasing the nonlinear parameter $e$ slightly not only changes the model because it is more eccentric, but also because at this new value of $e$ the linear parameters have different values. These complications can be avoided if one uses the linear basis of the orbits presented here in an ordi-

\footnote{At least 1/4 of known planetary systems show evidence of multiple companions (Wright et al. 2007)}
nary, nonlinear 5\(n + 1\) parameter fit, but the convergence and speed-up benefits will not be as great.

The sections below describe a general method of calculating an \(n\)-planet Keplerian radial velocity model given values for the \(3n\) nonlinear parameters, and of calculating the derivatives of that model with respect to those parameters. These equations can be used in a user-defined function for input to an LM minimization code, or a brute-force \(\chi^2\) map.

2.2. Linear Parameters in the Radial Velocity Problem

We wish to find the parameters of the \(n\)-planet model

\[
u(t) = \sum_{j=1}^{n} [K_j \cos(\omega_j + f_j(t)) + e_j \cos \omega_j] + \gamma + d \cdot (t - t_0)
\]

which best fits the set of observed radial velocities \(\vec{v}\), measured at times \(\vec{t}\), with uncertainties \(\sigma\) in a least-squares sense. Here, \(K_j\) and \(\omega_j\) are the usual Keplerian parameters for planet \(j\), \(\gamma\) is the time-independent velocity offset, \(d\) is the trend parameter, \(f_j(t)\) is the true anomaly of planet \(j\) at time \(t\), and \(t_0\) is a conveniently chosen epoch of the observations. The true anomaly is defined implicitly in terms of the other three Keplerian parameters \((P_j, t_p, j, e_j)\) through the relations

\[
\tan \frac{f_j(t)}{2} = \frac{1 + e_j}{1 - e_j} \tan \frac{E_j(t)}{2}
\]

\[
E_j(t) - e_j \sin E_j(t) = \frac{2\pi(t - t_p)}{P_j} = M_j(t)
\]

In Eq. 3 (known as Kepler’s Equation), \(E_j\) is called the eccentric anomaly of planet \(j\), and \(M\) is known as the mean anomaly.\(^7\)

We identify the linear parameters by rewriting Eq. 1 as

\[
u(t) = \sum_{j=1}^{n} [h_j \cos f_j(t) + c_j \sin f_j(t)] + v_0 + d \cdot (t - t_0)
\]

where

\[
h_j = K_j \cos \omega_j
\]

\[
c_j = -K_j \sin \omega_j
\]

and

\[
v_0 = \gamma + \sum_{j=1}^{n} K_j e_j \cos \omega_j
\]

These linear parameters can be converted back to Keplerian orbital elements through the relations

\[
K_j = \sqrt{h_j^2 + c_j^2}
\]

\[
\tan \omega_j = -\frac{c_j}{h_j}
\]

(where \(\omega_j\) is chosen so that \(\sin \omega_j\) has the sign of the numerator) and

\[
\gamma = v_0 - \sum_{j=1}^{n} K_j e_j \cos \omega_j
\]

The masses of the orbiting planets can be inferred from their corresponding semi-amplitudes \(K_j\), defined for a single-planet system with planet mass \(m\) and stellar mass \(m_*\) as

\[
K^3 = \frac{2\pi G}{P(1 - e^2)^2} \left( \frac{m^3 \sin^3 \iota}{(m_* + m)^2} \right)
\]

where \(G\) is Newton’s gravitational constant. The fraction in parentheses is known as the mass function of the system.

The problem can now be divided into two parts: an algorithmic search through parameter space for the best-fit nonlinear parameters \(P_j, e_j,\) and \(t_{p,j}\) with a computer routine such as an LM or an MCMC code, and at each step in that search an analytic solution for the linear parameters that fit best there.

2.3. Solving for the Linear Parameters

Given a set of values for the nonlinear parameters, we can fit for the linear parameters in Eq. 4 through \(\chi^2\) minimization. We denote the row vector of linear parameters \(\vec{\beta}\) as

\[
\vec{\beta} = \{h_1, c_1, h_2, c_2, \ldots, h_n, c_n, v_0, d\}
\]

We define \(\chi^2\) the usual way:

\[
\chi^2 = \sum_{k=1}^{N} \frac{(v_k - u(t_k))^2}{\sigma_k^2}
\]

and minimize it with respect to each of the linear parameters in \(\vec{\beta}\) simultaneously:

\[
\frac{\partial \chi^2}{\partial \beta_i} = -2 \sum_{k=1}^{N} \frac{v_k - u(t_k)}{\sigma_k^2} \frac{\partial u}{\partial \beta_i} \bigg|_{t=t_k} = 0
\]

We can express this more compactly by invoking matrix algebra. For the problem of Keplerian orbits, we define the matrix \(\vec{F}\) as

\[
\vec{F} = \begin{bmatrix}
\cos f_{1,1} & \cos f_{1,2} & \ldots \\
\sin f_{1,1} & \sin f_{1,2} & \ldots \\
\cos f_{2,1} & \cos f_{2,2} & \ldots \\
\vdots & \vdots & \ddots \\
\cos f_{n,1} & \cos f_{n,2} & \ldots \\
\sin f_{n,1} & \sin f_{n,2} & \ldots \\
1 & 1 & \ldots \\
\end{bmatrix}
\]

where \(f_{j,k} = f_j(t_k)\). This allows us to write the model velocities at times \(\vec{t}\) (Eq. 1) as

\[
\vec{u} = \vec{\beta} \vec{F}
\]

We also define the diagonal weight matrix \(\vec{W}\) such that

\[
W_{kl} = \delta_{kl}/\sigma_k^2
\]

where we have used the Kronecker delta symbol. We can then write the system of equations in Eq. 14 as

\[
\frac{\partial \chi^2}{\partial \vec{\beta}} = -2(\vec{v} - \vec{\beta} \vec{F} \vec{W} \vec{F}^T) = 0
\]

Solving for \(\vec{\beta}\) we have

\[
\vec{\beta} = \vec{v} \vec{W} \vec{F}^T \vec{e}
\]
where we have denoted the error matrix (also called the variance-covariance matrix)
\[
e = (FWF^T)^{-1}
\]
(20)

Eq. 19 represents the general solution to the linear least-squares problem for an appropriately defined $F$. In the context of Keplerian fits, given a set of nonlinear parameters $P$, $e$, and $t_p$ for each planet, the remaining Keplerian orbital elements can be found from $\vec{\beta}$ using Eqs. 8–10.

2.4. Calculation of Explicit (Analytic) Derivatives for Use in the LM Algorithm

The derivative of the model velocities $\vec{u}$ with respect to any nonlinear parameter can be found from Eq. 16:
\[
\frac{d\vec{u}}{dx} = \frac{d\vec{\beta}}{dx} F + \vec{\beta} \frac{dF}{dx}
\]
(21)

where $x$ stands for any of the nonlinear parameters (here, $P_j$, $e_j$, or $t_{p,j}$).

From Eq. 19 we have
\[
\frac{d\vec{\beta}}{dx} = \vec{v} W \left( \frac{d(F^T)}{dx} \epsilon + F^T \frac{d\epsilon}{dx} \right)
\]
(22)

From the definition of a matrix inverse we can express the last derivative as
\[
\frac{d\epsilon}{dx} = -\epsilon \left( \frac{d}{dx} \epsilon^{-1} \right) \epsilon
\]
(23)

Using Eq. 20 we then have
\[
\frac{d\epsilon}{dx} = -\epsilon \frac{d}{dx} (FWF^T) \epsilon = -\epsilon \left( \left( \frac{dF}{dx} W F^T \right) + \left( \frac{dF}{dx} F^T \right)^T \right) \epsilon
\]
(25)

Eqs. 16–24 are not specific to the RV Kepler problem, but are a general method of calculating explicit derivatives in a model with both linear and nonlinear parameters, and so can be applied to any analogous problem. For instance, the problem of fitting an orbit using astrometric data also has linear parameters as we show in §3.

2.5. Explicit Derivatives for the Radial Velocity Model

For the case of a Keplerian RV model, from Eq. 15 we have
\[
\frac{dF}{dx} = \begin{bmatrix}
-\sin f_{1,1} f'_{1,1} & -\sin f_{1,2} f'_{1,2} & \cdots \\
\cos f_{1,1} f'_{1,1} & \cos f_{1,2} f'_{1,2} & \cdots \\
-\sin f_{2,1} f'_{2,1} & -\sin f_{2,2} f'_{2,2} & \cdots \\
\cos f_{2,1} f'_{2,1} & \cos f_{2,2} f'_{2,2} & \cdots \\
\vdots & \vdots & \ddots \\
-\sin f_{n,1} f'_{n,1} & -\sin f_{n,2} f'_{n,2} & \cdots \\
\cos f_{n,1} f'_{n,1} & \cos f_{n,2} f'_{n,2} & \cdots \\
0 & 0 & 0 & \cdots 
\end{bmatrix}
\]
(26)

where $f'_{j,k}$ is $d f_{j,k} / dx$. Note that since $f_{1,k}$ refers to the true anomaly of planet $j$, it vanishes when $x$ refers to a parameter of a different planet (e.g., $d f_{j,k} / d P_l = 0$ when $j \neq l$). This means that the matrix $dF/dx$ has only two nonzero rows. We can therefore suppress subscripts below for clarity.

We can calculate the nonzero derivatives as
\[
\frac{df}{dx} = \frac{df}{dx} + \frac{df}{dx} \frac{dE}{dE} \frac{dx}{dx}
\]
(27)

where, from Kepler’s Equation (Eq. 3), we have
\[
\frac{dE}{dP} = \frac{-2\pi(t - t_p)/P^2}{1 - \epsilon \cos E}
\]
(28)
\[
\frac{dE}{dt_p} = \frac{-2\pi/P}{1 - \epsilon \cos E}
\]
(29)
\[
\frac{dE}{de} = \frac{\sin E}{1 - \epsilon \cos E}
\]
(30)

and from Eq. 2 we have
\[
\frac{\partial f}{\partial E} = \sqrt{1 + e^2 + \cos f / (1 - e^2)}
\]
(31)

From Eq. 2 we can also write
\[
\frac{\partial f}{\partial e} = \frac{2 \tan(E/2) \cos^2(f/2)}{(1 - e)\sqrt{1 - e^2}}
\]
(32)

but it is more computationally convenient to note that this happens to simplify to
\[
\frac{\partial f}{\partial e} = \frac{\partial f}{\partial E} \frac{\sin E}{1 - e^2}
\]
(33)

Finally,
\[
\frac{\partial f}{\partial t_p} = \frac{\partial f}{\partial t_p} = 0
\]
(34)

Eqs. 22–34 can be used to calculate the terms in Eq. 21, yielding the explicit derivatives used by LM method fitting routines, such as MPFIT.

2.6. Variations on the n-Planet RV Model

In the context of the Kepler problem, the above equations include a parameter for a linear trend in the data. In practice, fitting for such a trend will only occasionally be necessary. When not needed the $d$ parameter in $\vec{\beta}$ and the bottom rows of the $F$ and $dF/\partial x$ matrices can simply be left out of the calculations. Similarly, the trend parameter can effectively be held fixed at a nonzero value by simply subtracting the desired value from the data before fitting.

These matrices can also be easily extended to handle the case of combining data from multiple telescopes between which there exist RV offsets. This is accomplished by extending the data vectors $\vec{v}$, $\vec{t}$, and $\vec{d}$ to include data from all telescopes, and extending the vector $\vec{\beta}$ to include a separate offset parameter for each telescope after the first. The corresponding rows of $F$ must then be filled with 1’s in those columns corresponding to data from the appropriate telescope, and 0’s elsewhere. Naturally, in $dF/\partial x$ the elements of these rows are all 0.

3. APPLICATIONS TO ASTROMETRY

3.1. Astrometry Alone

The above method can also be extended to the problem of fitting for the Keplerian elements of an orbit from astrometric data. Here we present a simplified model of
μas astrometric data of the sort that might be provided by a space observatory such as NASA’s Space Interferometry Mission (SIM Lite). We anticipate modifying our algorithms for a more realistic model of interferometric data and its noise sources (Sozzetti 2005; Eriksson & Lindgren 2007; Catanzarite, Law, & Shao 2008) in the near future.

The linear basis for astrometric coordinates are the Thiele-Innes constants (e.g. Binnendijk 1960), and are well documented as useful tools for astrometric curve-fitting (e.g. Casertano et al. 2008). The astrometric perturbations caused by an orbiting companion can be described in terms of six astrometric orbital elements: in addition to $e$ and $t_p$, we have $a$, the semi-major axis of the star’s apparent orbit on the sky in units of arc; $\Omega$, the longitude of the ascending (approaching) node (measured as a position angle on the sky): $i$, the inclination of the orbit on the sky (such that $i = 0$ corresponds to a face-on, clockwise orbit); and $\omega_s$, the longitude of periastron of the star’s orbit.

The Thiele-Innes constants are defined in terms four of the astrometric elements of the star’s orbit about the secondary:

$$A = a(\cos \Omega \cos \omega_s - \sin \Omega \sin \omega_s \cos i)$$
$$B = a(\cos \Omega \cos \omega_s + \sin \Omega \sin \omega_s \cos i)$$
$$F = a(-\cos \Omega \sin \omega_s - \sin \Omega \cos \omega_s \cos i)$$
$$G = a(-\sin \Omega \sin \omega_s - \cos \Omega \cos \omega_s \cos i)$$
$$C = a \sin \omega_s \sin i$$
$$H = a \cos \omega_s \sin i$$

These constants can be quickly computed using rotation matrices:

$$\begin{bmatrix} A \\ F \\ B \\ G \\ C \\ H \end{bmatrix} = aR_z(\omega_s)R_x(i)R_z(\Omega)$$

where $R$ is the 3-D rotation matrix

$$R_z(\Omega) = \begin{bmatrix} \cos \Omega & \sin \Omega & 0 \\ -\sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\omega_s$ is the argument of periastron of the orbit of the star.

We can transform the Thiele-Innes constants back to Keplerian orbital elements of the planet with the relations:

$$\tan(\omega_s + \Omega) = \frac{B - F}{A + G}$$
$$\tan(\omega_s - \Omega) = \frac{-B + F}{A - G}$$
$$\tan^2\left(\frac{i}{2}\right) = \frac{(A - G)\cos(\omega_s + \Omega)}{(A + G)\cos(\omega_s - \Omega)}$$
$$a = \frac{(A \cos \omega_s - F \sin \omega_s) \cos \Omega - (A \sin \omega_s + F \cos \omega_s) \sin \Omega}{\sin i}$$
$$\omega = \omega_s + \pi$$

where the quadrants of $\omega_s - \Omega$ and $\omega_s + \Omega$ are determined by the signs of the numerators in Eqs. 43 & 44. These relations leave a $\pm \pi$ ambiguity in $\omega_s$, $\omega$, and $\Omega$ that can only be resolved by radial velocities, without which convention dictates that we choose the solution with $\Omega < \pi$.

The Thiele-Innes constants $C$ and $H$ are closely related to the $e$ and $h$ constants of Eqs. 5 & 6. The set of six constants have the identity

$$a^2 = A^2 + B^2 + C^2 = F^2 + G^2 + H^2$$

In astrometry there are five parameters that describe a star’s motion in the absence of orbiting companions: $\Delta a_0 \cos \delta$ and $\Delta a_0$, the difference between the true and nominal position of the system at $t_0$; $\mu_\alpha$ and $\mu_\delta$, the proper motions in the RA and Dec directions; and $\varpi$, the parallax of the system. Our model for the astrometric displacement of a star due to parallax, proper motion, and a system of unseen planets in terms of the Thiele-Innes constants at times $\tau$ is:

$$\Delta \delta_k = \sum_{j=1}^n [A_j X_{j,k} + F_j Y_{j,k}] + \Delta a_0 + \varpi \Pi_{j,k} + \mu_\delta (\tau_k - t_0)$$
$$\Delta a_k \cos \delta = \sum_{j=1}^n [B_j X_{j,k} + G_j Y_{j,k}] + \Delta a_0 \cos \delta + \varpi \Pi_{j,k} + \mu_\alpha (\tau_k - t_0)$$

where $X$ and $Y$ are the so-called elliptical rectangular coordinates, defined as

$$X_{j,k} = \cos E_j(\tau_k) - e_j$$
$$Y_{j,k} = \sqrt{1 - e_j^2} \sin E_j(\tau_k)$$

where $E_j$ is the eccentric anomaly and the quantities $\Pi_{\alpha,k}$ and $\Pi_{\delta,k}$ refer to the astrometric displacements due to parallax in the $\alpha$ and $\delta$ directions during observation $k$, which are given by (cite Supplement to Astronomical Almanac here):

$$\Pi_{\alpha,k} = r_x(\tau_k) \sin \alpha - r_y(\tau_k) \cos \alpha$$
$$\Pi_{\delta,k} = (r_x(\tau_k) \cos \alpha + r_y(\tau_k) \sin \alpha) \sin \delta - r_z(\tau_k) \cos \delta$$

Here $(r_x, r_y, r_z)$ represent the Cartesian components in equatorial coordinates of the position of the observatory, $\tau'$, at time $\tau$ with respect to the Solar System barycenter (in units of AU when $\varpi$ is in arcsec). These values for the Earth are available from the NASA Jet Propulsion Laboratory Solar System ephemerides, but for $\mu$ as and spaceborne work the precise position of the observatory itself is required.

Note that since $a$ is the apparent semi-major axis of the star’s orbit in units of arc, its relationship to the mass of the secondary depends on the method of astrometry used. For astrometric perturbations due to an unseen planet, (that is, absolute astrometric displacements with respect to the sidereal frame, as measured by SIM Lite) we have from Kepler’s Third Law

$$a^3 = \frac{\varpi^3 m^3}{(m_s + m)^2} P^2$$

where $a$ is measured in arcseconds when $P$ is measured in years, $\varpi$ is the parallax in arcseconds, and $m$ is the

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8 The orbital parameters of the star and those of the unseen companion are all identical except $a$, which differs by a factor of $m/m_s$, and $\omega$, which differs by $\pi$.

9 In this paper the bare symbols $\alpha$ and $\delta$ will always refer to the nominal right ascension and declination of a star at $t_0$, absent the effects of parallax and astrometric displacement from companions.

10 http://ssd.jpl.nasa.gov
mass of the unseen companion and \( m_* \) is the mass of the star in solar masses.\(^{11}\)

We can now extend Eq.16 to the case of 2-D data by defining our vector of \( N \) measurements taken at times \( \tau \):

\[
[\Delta \delta_1, \Delta \delta_2 \ldots \Delta \delta_N, \Delta \alpha_1 \cos \delta, \Delta \alpha_2 \cos \delta \ldots \Delta \alpha_N \cos \delta]
\]

and our model with linear parameters for \( n \) planets:

\[
\beta = [A_1, B_1, F_1, G_1 \ldots A_n, B_n, F_n, G_n, \Delta \delta_0, \Delta \alpha_0 \cos \delta, \mu \delta, \mu \alpha, \omega]
\]

(57)

and matrix \( \mathbf{F} = \)

\[
\begin{bmatrix}
X_{1,1} & X_{1,2} & \cdots & X_{1,N} & 0 & 0 & \cdots & 0 \\
Y_{1,1} & Y_{1,2} & \cdots & Y_{1,N} & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & Y_{1,1} & Y_{1,2} & \cdots & Y_{1,N} \\
0 & 0 & \cdots & 0 & X_{2,1} & X_{2,2} & \cdots & X_{2,N} \\
0 & 0 & \cdots & 0 & Y_{2,1} & Y_{2,2} & \cdots & Y_{2,N} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & X_{n,1} & X_{n,2} & \cdots & X_{n,N} \\
0 & 0 & \cdots & 0 & Y_{n,1} & Y_{n,2} & \cdots & Y_{n,N} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\tau_1 - t_0 & \tau_2 - t_0 & \cdots & \tau_N - t_0 & 0 & 0 & \cdots & 0 \\
\Pi_{\delta,1} & \Pi_{\delta,2} & \cdots & \Pi_{\delta,N} & \Pi_{\alpha,1} & \Pi_{\alpha,2} & \cdots & \Pi_{\alpha,N}
\end{bmatrix}
\]

The nonzero components of \( d\mathbf{F}/d\tau \) can be calculated from:

\[
\begin{align*}
\frac{dX}{d\tau} &= -\frac{dE}{dP} \sin E \\
\frac{dY}{d\tau} &= -\frac{dE}{dP} \sin E \\
\frac{dX}{dt_p} &= -\frac{dE}{de} \sin E - 1 \\
\frac{dY}{dt_p} &= \sqrt{1 - e^2} \cos E \frac{dE}{de} \\
\frac{dV}{dt_p} &= \sqrt{1 - e^2} \cos E \frac{dE}{de} \\
\frac{dY}{de} &= \sqrt{1 - e^2} \cos E \frac{dE}{de} - \frac{e \sin E}{\sqrt{1 - e^2}}
\end{align*}
\]

(58)

and Eqs. 28–30.

### 3.2. Astrometry in Arbitrary Coordinates

Astrometry does not always deliver contemporaneous (\( \Delta \alpha \cos \delta, \Delta \delta \)) pairs at a common time \( t \). In the general case, a baseline determines the 1-D displacement of a star from some reference at an intermediate position angle on the sky (i.e., not necessarily 0 [as in the case for \( \Delta \delta \) or \( \pi/2 \) [as for \( \Delta \alpha \cos \delta \)]). We can combine Eqs. 49 & 50 to a

\[ a^2 = \frac{C_j^2 + H_j^2}{\sin^2 i_j} \]

(60)

\[ \tan \omega_j = -\frac{C_j}{H_j} \]

(61)

where \( \omega_j \) is chosen so \( \sin \omega_j \) has the same sign as \( -C_j \).

### 3.3. Combining Astrometry with Radial Velocities

Combining astrometric data with radial velocity data is not as simple as combining the RV-only and astrometry-only schemes outlined above, because the six linear parameters (\( A, B, F, G, \) and \( c \) and \( h \)) are a combination of only five Keplerian elements (\( K, a, \Omega, \omega, \) and \( i \)), and the problem is thus overconstrained. One solution would be to minimize \( \chi^2 \) subject to the appropriate constraints using Lagrange multipliers, but the resulting set of nonlinear equations may not be guaranteed to have a unique solution and would be difficult to solve for the case of an arbitrary number of planets.

Another solution is to adapt the \( C \) and \( H \) constants to accommodate general astrometric data, and these constants are closely related to the scheme for RV data used in §2.2. To do this, we define our vector of measurements to be

\[
[i, \rho \eta]
\]

(62)
where the velocities are taken at times $\tilde{t}$ and the astrometry at times $\tilde{\tau}$, and define our model, as in Eq. 16:

$$\tilde{\beta} = [H_1, C_1, H_2, C_2, \ldots H_n, C_n, v_0, d, \Delta \delta_0, \Delta \alpha_0, \cos \delta, \mu_0, \mu_\alpha, \varpi]$$

(71)

From Eqs. 5, 6, 41, & 55 we have:

$$h_j = -\lambda_j H_j$$

(72)

$$c_j = \lambda_j C_j$$

(73)

where we have introduced a $\lambda$, a combination of nonlinear orbital parameters of planet $j$ which has units of velocity and has the value

$$\lambda_j = \frac{2\pi A U}{\tilde{\omega} P_j \sqrt{1 - e_j^2}}$$

(74)

when the estimated parallax, $\tilde{\omega}$, is expressed in arcseconds.

The parameters $H_j$, $C_j$, and $v_0$ can be transformed into $m_j^2/(m_* + m_j)^2$, $\omega_j$, and $\gamma$ with Eqs. 8–11 & 72–73.

The appearance of $\omega$ in the definition of $\lambda_j$ (and therefore in $C_j$ and $H_j$) indicates a fundamental nonlinearity in the combined RV-astrometry problem: the parallax is not a truly linear parameter. However, because it is nearly linear, if a good estimate of the parallax is available then this can be used in the formula for $\lambda_j$ (we indicate the approximate nature of the parallactic term with a tilde). Once the parallax is solved for more precisely, we will use it again to deal with other, smaller nonlinear terms in §4.

The first columns of the $F$ matrix, corresponding to the radial velocity measurements, now read

$$
\begin{bmatrix}
-\cos f_{1,1} \lambda_1 & -\cos f_{1,2} \lambda_1 & \ldots \\
\sin f_{1,1} \lambda_1 & \sin f_{1,2} \lambda_1 & \ldots \\
-\cos f_{2,1} \lambda_2 & -\cos f_{2,2} \lambda_2 & \ldots \\
\sin f_{2,1} \lambda_2 & \sin f_{2,2} \lambda_2 & \ldots \\
\vdots & \vdots & \ddots \\
-\cos f_{n,1} \lambda_n & -\cos f_{n,2} \lambda_n & \ldots \\
\sin f_{n,1} \lambda_n & \sin f_{n,2} \lambda_n & \ldots \\
1 & 1 & \ldots \\
t_1 - t_0 & t_2 - t_0 & \ldots \\
0 & 0 & \ldots \\
0 & 0 & \ldots \\
0 & 0 & \ldots \\
0 & 0 & \ldots \\
0 & 0 & \ldots \\
\end{bmatrix}
$$

(75)

and the rest of the columns, corresponding to the astrometric measurements, read

$$
\begin{bmatrix}
S_{1,1} & S_{1,2} & \ldots \\
T_{1,1} & T_{1,2} & \ldots \\
S_{2,1} & S_{2,2} & \ldots \\
T_{2,1} & T_{2,2} & \ldots \\
\vdots & \vdots & \ddots \\
S_{n,1} & S_{n,2} & \ldots \\
T_{n,1} & T_{n,2} & \ldots \\
0 & 0 & \ldots \\
0 & 0 & \ldots \\
\cos \theta_1 & \cos \theta_2 & \ldots \\
\sin \theta_1 & \sin \theta_2 & \ldots \\
(\tau_1 - t_0) \cos \theta_1 & (\tau_2 - t_0) \cos \theta_2 & \ldots \\
(\tau_1 - t_0) \sin \theta_1 & (\tau_2 - t_0) \sin \theta_2 & \ldots \\
\Pi_{\delta,1} \cos \theta_1 + \Pi_{\alpha,1} \sin \theta_1 & \Pi_{\delta,2} \cos \theta_2 + \Pi_{\alpha,2} \sin \theta_2 & \ldots
\end{bmatrix}
$$

(76)

For the nonlinear parameters $P_j$, $T_{p,j}$, and $e_j$, the two nonzero rows of $d\mathbf{F}/dx$ can be calculated from:

$$
\frac{d}{dx} \left( \cos f(t) \lambda \right) = -\sin f(t) f'(t) \lambda + \cos f(t) \frac{d\lambda}{dx}
$$

(77)

$$
\frac{d}{dx} \left( -\sin f(t) \lambda \right) = -\cos f(t) f'(t) \lambda - \sin f(t) \frac{d\lambda}{dx}
$$

(78)

where

$$
\frac{d\lambda}{dP} = -\lambda \\
\frac{d\lambda}{de} = -\frac{\lambda}{1 - e^2}
$$

(79)

(80)

and

$$
\begin{bmatrix}
\frac{dS}{dx} \\
\frac{dT}{dx}
\end{bmatrix} =
\begin{bmatrix}
\cos(\Omega - \theta) \csc i & -\sin(\Omega - \theta) \cot i \\
-\sin(\Omega - \theta) \cot i & -\cos(\Omega - \theta) \csc i
\end{bmatrix}
\begin{bmatrix}
\frac{dX}{dx} \\
\frac{dY}{dx}
\end{bmatrix}
$$

(82)

and Eqs. 27–34, 59–64. Now we have introduced two additional nonlinear parameters, $\Omega$ and $i$.

$$
\begin{bmatrix}
\frac{dS}{d\Omega} \\
\frac{dT}{d\Omega}
\end{bmatrix} =
\begin{bmatrix}
-\sin(\Omega - \theta) \csc i & -\cos(\Omega - \theta) \cot i \\
-\cos(\Omega - \theta) \cot i & \sin(\Omega - \theta) \csc i
\end{bmatrix}
\begin{bmatrix}
X \\
Y
\end{bmatrix}
$$

(83)

and

$$
\begin{bmatrix}
\frac{dS}{di} \\
\frac{dT}{di}
\end{bmatrix} =
\begin{bmatrix}
-\sin(\Omega - \theta) \csc i \cot i & \sin(\Omega - \theta) \csc^2 i \\
\sin(\Omega - \theta) \csc^2 i & \cos(\Omega - \theta) \csc i \cot i
\end{bmatrix}
\begin{bmatrix}
X \\
Y
\end{bmatrix}
$$

(84)

4. NONLINEAR TERMS

4.1. Sources of Nonlinearity

There are several small nonlinear terms which are important at the m/s and μas level, especially for the
nearby and high-proper motion stars likely to be observed by SIM Lite.

Secular acceleration — A star with significant proper motion will have the radial component of its space velocity change with position on the sky, resulting in a secular change in the radial velocity up to ~1 m/s/yr for the most extreme cases. Secular acceleration is given, to first order, by:

\[ \dot{v}_r = D\mu^2 \]  

(85)

where, \( v_r \) is the bulk radial velocity of the star, \( D \) is the star’s distance and \( \mu \) is the total proper motion in radians per unit time. This term will be absorbed into the linear parameter \( d \), if present, and so could be ignored.

Parallax changes — The change in parallax of nearby stars due to their radial velocity may be of order 0.3 \( \mu \)as/yr. The shape of the parallactic motion is also a function of position on the sky, and thus of the proper motion. These changes can be of order 5 \( \mu \)as/yr. The radial velocity term is given by:

\[ \dot{\varpi} = -\varpi v_r/D \]  

(86)

Proper motion changes — The flip side of secular acceleration is proper motion change due to change in distance. This effect can be of order 3 \( \mu \)as/yr and is given by:

\[ \dot{\vec{\mu}} = -\vec{\mu} v_r/D \]  

(87)

Curvilinear effects — The curvilinear nature of spherical coordinates can produce what are essentially nonlinear terms depending on how astrometric displacements are defined. These effects are of the same order as the above proper motion changes, and more pronounced near the poles. For \( \mu \)as astrometry, it suffices to handle these effects by employing a rectilinear grid:

In this work, astrometric displacements labeled \( \Delta \alpha \cos \delta \) and \( \Delta \delta \) do not strictly refer to changes in the right ascension and declination of the star, but refer to displacements along rectilinear axes aligned along those dimensions at the nominal position of the star. That is, if the unit vector pointing to the nominal position of the star from the Solar System barycenter is defined:

\[ \hat{p} = [\cos \alpha \cos \delta, \sin \alpha \cos \delta, \sin \delta] \]  

(88)

then the unit vectors pointing east and north are given by

\[ \hat{\alpha} = [0, 0, 1] \times \hat{p} \]  

(89)

\[ \hat{\delta} = \hat{p} \times \hat{\alpha} \]  

(90)

and the astrometric displacements are given by

\[ \Delta \delta = \hat{p}' \cdot \hat{\delta} \]  

(91)

\[ \Delta \alpha \cos \delta = \hat{p}' \cdot \hat{\alpha} \]  

(92)

where \( \hat{p}' \) is the displaced position of the star and \( \hat{\delta} \) and \( \hat{\alpha} \) are constant.

Interferometric cross terms — The 1-D interferometric measurement of astrometric displacement on the sky may be complicated by the motion of the reference and target stars. That is, the calculation of \( \theta \) in Eq. 65 may require proper motion advanced or parallax corrected positions in a manner specific to the details of a particular instrument’s measurement of \( \theta \). These cross terms are likely to be small, and so they can be estimated and refined in the same manner as the other nonlinear terms, if necessary.

RV-astrometry cross terms — The RV semi-amplitude \( K \) is related to the astrometric semi-major axis, \( a \), by the parallax, and so \( \varpi \) is not strictly a linear parameter in the combined astrometry-RV problem. This effect can be large if the parallax is small or not known, and should not be ignored for any system.

Relativistic Terms — Gravitational deflection by Solar System objects and relativistic stellar aberration produce large, time dependent astrometric displacements that depend on the position of the star, and are therefore slightly nonlinear. Because these displacements can be calculated to better than \( \mu \)as precision given an estimate of the star’s true position to arcsecond precision, these effects are ignored here.

4.2. Incorporating Nonlinear Terms

Implementing these small nonlinear effects in our model requires good estimates of the astrometric parameters (when astrometric data is available, these parameters can be estimated from a first-pass solution assuming no planetary companions). The system is then solved using these estimates to calculate the second-order terms above. Below, we indicate these estimated astrometric parameters (when astrometric data is available, these parameters can be estimated from a first-pass solution assuming no planetary companions). The system is then solved using these estimates to calculate the second-order terms above. Below, we indicate these estimated astrometric parameters with a tilde to distinguish them from the above proper motion changes, and more pronounced near the poles.

We can include these nonlinear terms by making the following substitutions to the bottom 3 rows of the matrix \( F \) in Eqs. 58 & 76:

\[ \tau_k - t_0 \rightarrow \left(1 - (\tau_k - t_0)v_r/(2D)\right)(\tau_k - t_0) \]  

(93)

\[ \Pi_{\alpha,k} \rightarrow 1 - (\tau_k - t_0)\frac{v_r}{D}\Pi_{\alpha,k} \]  

(94)

\[ \Pi_{\delta,k} \rightarrow 1 - (\tau_k - t_0)\frac{v_r}{D}\Pi_{\delta,k} \]  

(95)

and in the definitions of \( \Pi_k \) (Eqs. 53 & 54):

\[ \Pi_{\alpha,k} \rightarrow \Pi_{\alpha,k} + (\Delta \alpha_0 \cos \delta + (\tau_k - t_0)\mu_\alpha)\hat{\alpha} \cdot \hat{p} \]  

(96)

\[ \Pi_{\delta,k} \rightarrow \Pi_{\delta,k} + (\Delta \delta_0 + (\tau_k - t_0)\mu_\delta)\hat{\delta} \cdot \hat{p} \]  

(97)

where the quantities in parentheses have units of radians. The estimated terms \( \Delta \alpha_0 \cos \delta \) and \( \Delta \delta_0 \) will likely be zero at first, but may be iteratively refined with the other estimated parameters.

Finally, the secular acceleration can be accommodated by simply subtracting off the appropriate, approximate linear trend to the RV data, or, if the trend parameter is present, allowing it to be absorbed in \( d \).

This procedure of estimating and refining linear astrometric terms will also work for the nonlinear term introduced by the appearance of \( \varpi \) in \( H \) and \( C \) in Eqs. 72 &
Alternatively, the parallax can be treated as a nonlinear parameter from the outset and eliminated from the linear coefficients in $\vec{\beta}$ entirely.

5. IMPLEMENTATION

5.1. Public code

We have implemented these algorithms into a set of IDL software routines which we have made available publicly for the fitting of radial velocity data.\(^\text{14}\) The software package includes RVLIN, the user-defined function that may be passed to MPFIT, and RV_FIT_MP, a “wrapper” routine that employs MPFIT to fit a multi-planet Keplerian model to a user-supplied set of radial velocity data. We anticipate maintaining and improving this package, and eventually incorporating astrometric and transit data analysis. This code, or components of it, are currently used by members of the California Planet Search. Below, we discuss our software package’s performance.

5.2. Speed-up From the Use of Explicit Derivatives

The use of explicit derivatives above in an LM code significantly speeds up the algorithm by avoiding unnecessary calculation of numerical derivatives. To quantify this improvement, we tested two cases with published RV data, the 2-planet system HD 217107 (Vogt et al. 2005) and the 5-planet system 55 Cnc (Fischer et al. 2008) (in both cases assuming no jitter). We combined the Lick and Keck RV data sets for both cases, solving for all $5n + 1$ parameters with carefully chosen step-sizes for numerical derivatives.\(^\text{16}\)

Using the same hardware described above, we again fit the Fischer et al. (2008) 55 Cnc RV data from two telescopes. We drew the initial guesses for every parameter from normal distributions centered on the Keplerian best-fit values and with width given by $s\sigma_x$, where $\sigma_x$ is the parameter uncertainty quoted in Fischer et al. (2008) and $s$ is a scale factor. We varied $s$ smoothly from 0 (where the initial guesses were the best-fit values exactly) to 10 (where every parameter is independently, randomly altered by $10\sigma_x$) over 4000 trials.

We did not attempt to fit for the telescope offset at each trial, as this particular routine was not optimized for such a task. We also did not vary the $\gamma$ parameter from its best-fit value (though we did fit for it), as no uncertainties were given for $\gamma$ in Fischer et al. (2008).

We compared the results of this custom code to our LM code employing linear parameters (§5.1) with both explicit and numerical derivatives. In some sense, this is not a fair test since the linear parameter models do not require or accept initial guesses for $K$ and $\omega$, and thus the initial guesses are closer to the best-fit values (since they differ in only 15 dimensions space, whereas the guesses for the nonlinear routine they differ in 25 dimensions.) We thus also ran an additional test, where the fully-nonlinear routine was provided the best-fit values for $K$ and $\omega$ for each planet, and thus had only 15 parameters varied (but still had all 26 parameters to fit). With this exception, we provided the same initial guesses to each of the four schemes in each trial.

We recorded the final $\chi^2$ reported by each routine for each trial and compared this with $\chi^2_{\text{min}}$, the best-fit value. We deemed any trial for which $(\chi^2 - \chi^2_{\text{min}}) < 2$ to be a “successful” convergence on the correct parameters.\(^\text{17}\) At each tested value of $s$, we weighted the full set of trials by a Gaussian window 75 trials wide (corresponding to $\sim 0.19\sigma$) and calculated the fraction on trials that were successful.

In Figure 1 we plot the fraction of successful tests as a function of $s$. The use of numerical derivatives had no overall effect on the convergence of the linear parameter routine, but was slower, on average, by a factor of 4.

Even with initial guesses $10\sigma_x$ from the best-fit values in all 15 parameters, the linear parameter routine found the global minimum of $\chi^2$ in roughly half of all trials. The fully-nonlinear routine, searching a 26-dimensional space, required much better initial guesses to achieve convergence. With the same 15 parameters varied, fits with initial parameters off by $3\sigma_x$ had only a 20% chance of properly converging. With all 25 orbital parameters varied, the initial guesses had to be within 1.5-$\sigma_x$ of their proper value to have a 50% chance of convergence. With 26 dimensions to search, there are many wrong paths for the LM algorithm to follow away from the global $\chi^2$ minimum.

6. CONCLUSIONS

\(^{16}\) That is to say, the only significant differences between the baseline code and the code described in § 5.1 is the exploitation of linear parameters.

\(^{17}\) $\chi^2_{\text{min}} \sim 2910$. Our results are only weakly sensitive to the precise definition of “successful”
Applying linear parameters to the problem of fitting Keplerian curves to radial velocity and astrometry data significantly improves the efficiency and reliability of multi-planet Keplerian fitting routines. This technique can be applied to many methods of searching this complex, nonlinear, multi-parameter $\chi^2$ space, including the Levenberg-Marquardt method, Markov chain Monte Carlo algorithms, and brute force approaches. Table 1 summarizes the various schemes used in this work.

We have identified the nonlinear terms relevant for radial velocity and astrometry data and shown how to incorporate these terms into a linear parameter scheme. In §2.4 we have provided analytic forms for explicit derivatives relevant to various applications of the Kepler problem.

In the case of RV-only data, use of explicit derivatives can speed up a fitting routine by a factor of 2–4, depending on the number of planets being fit. Use of linear parameters greatly improves the convergence properties of a multi-planet fitting routine. In the case of an actual 5-planet fit, a linear parameter model requires initial guesses within $10\sigma$ of their correct values in only 15 parameters to have a 50% chance of convergence, while a full, 26-parameter search requires all 26 parameters to be specified within $1.5-\sigma$.

The principle improvement from use of linear parameters comes from reducing the search space for an $n$-planet model. In the case of RV fitting, the reduction is from exploiting the linearity of $2n+1$ of the $5n+1$ fitted parameters, leaving only $3n$ nonlinear parameters to be fit algorithmically. The problem of fitting astrometric orbits can be similarly treated by exploiting the linearity of $4n+5$ of the $7n+5$ parameters, leaving, again, only $3n$ nonlinear parameters to be with a nonlinear fitting routine. When combining RV and astrometric data, as will be the case for SIM Lite, only $2n+6$ of the $7n+6$ model parameters are usefully linear under the scheme described here, leaving $5n$ nonlinear parameters.

We have derived a general expression for the explicit model derivatives for models employing linear parameters in nonlinear fits, appropriate for application in the LM method. This result is general and can be applied to problems beyond Keplerian fitting—indeed to any model with both linear and nonlinear parameters.

Eric Agol inspired §2 by pointing out how to exploit the linear parameters in the problem of RV fitting in the context of MPFIT. He has provided substantial guidance and ideas for this and future versions of our code. We thank him, Eric Ford, Matthew Metuspaugh, Tom Loredoo, and Alessandro Sozzetti for careful readings of this manuscript and their many substantive and constructive suggestions.

Our membership on a SIM Planet-Finding Astrometry Analysis Team (PI Matthew Metuspaugh) motivated §§3 & 4 of this work, and we are grateful to all of the Teams involved for their efforts and insights into the problem. We thank Jeff Valenti and Nikolai Piskunov for encouraging us to calculate the explicit derivatives for a general linear least-squares problem. Nikolai Piskunov provided insight into the sources of instability in the LM search algorithm. Matthew Metuspaugh provided guidance with the problem of fitting astrometric orbits and understanding the small astrometric cross terms. We thank Martin Sirk, Sam Halverson, and John Asher Johnson for their work employing and debugging various versions of our code. J.T.W received support from NSF grant AST-0504874, and A.W.H received support from NASA contract NAS7-03001 (JPL#1336910).

REFERENCES


Balan, S. T., & Lahav, O. 2008, ArXiv 0805.3532, 805


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Hastings, W. 1970, Biometrika, 57
Markwardt, C. B. 2009, ArXiv 0902.2859
## TABLE 1
SUMMARY OF LINEARIZATION SCHEMES

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