Astro 6531: Problem Set 3

Hand out: March 10; **Due: March 24, in class**

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1. Nonradial Oscillations of Stars:
(i) Consider nonradial oscillations of a nonrotating star with mass M and radius R. The wavefunctions assume the form $\delta P,\ \xi_r \propto Y_{lm}(\theta,\phi)$. Derive the boundary conditions as given in class:

$$\frac{d\delta\Phi}{dr} = \frac{l}{r}\delta\Phi, \quad \xi_r = \frac{l}{\omega^2 r}\left(\frac{\delta P}{\rho} + \delta\Phi\right), \qquad \text{for} \quad r \to 0, \tag{1}$$

and

$$\frac{d\delta\Phi}{dr} = -\frac{l+1}{r}\delta\Phi, \quad \delta P = \rho g \xi_r, \quad (\text{where } g = GM/R^2) \qquad \text{for} \quad r = R. \tag{2}$$

(ii) **Kelvin Modes:** A self-gravitating incompressible sphere has uniform density ρ, mass M and radius R. It undergoes nonradial oscillations. Show that the mode frequencies are given by

$$\omega^2 = A(l)\pi G\rho, \tag{3}$$

where $A(l)$ is a function of l (as in Y_{lm}). Derive the function $A(l)$.

2. Inertial Waves. An unbounded incompressible fluid is rotating with constant angular velocity $\mathbf{\Omega}$ and is subjected to a force $-\nabla\Phi$. Show that, in the rotating frame, a wave-like disturbance of the form $e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$ (with both $\mathbf{k}$ and ω real) satisfies the dispersion relation

$$\omega = \pm 2\mathbf{\Omega}\cdot\mathbf{k}/|\mathbf{k}|. \tag{4}$$

3. Viscous Dissipation Rate. Consider an incompressible fluid in the absence of external force (such as gravity). Show that the rate of change of kinetic energy of the fluid is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\frac{\partial}{\partial x_j}\left[\rho v_j\left(\frac{1}{2}v^2 + \frac{P}{\rho}\right) + v_i t_{ij}\right] + t_{ij}\frac{\partial v_i}{\partial x_j},$$

where t_{ij} is viscous stress tensor

$$t_{ij} = \rho\nu\sigma_{ij} = -\rho\nu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

Argue from this that the rate of energy dissipation per unit volume is $(1/2)\rho\nu\,\sigma_{ij}^2$.

4. Practice on Curvilinear Coordinate. Derive the incompressible Navier-Stokes equation in the cylindrical coordinates (R,ϕ,z). That is, complete the expressions:

$$\frac{\partial v_R}{\partial t} = \cdots, \quad \frac{\partial v_\phi}{\partial t} = \cdots, \quad \frac{\partial v_z}{\partial t} = \cdots.$$

The main goal of the exercise is to make sure you know how to do it. So if you find this a boring exercise, just do the R-component is adequate: $\partial v_R/\partial t = \cdots$. The answer can be found in many fluid mechanics texts.
Hint: You may do this whatever way you like. One possibility is to use the identity:

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v}) - \mathbf{v}\times(\nabla\times\mathbf{v}), \qquad \nabla^2\mathbf{v} = \nabla(\nabla\cdot\mathbf{v}) - \nabla\times(\nabla\times\mathbf{v}).$$