1. **(Epicyclic frequency)** Consider a star moving in a circular orbit at radius \( r = r_0 \) and \( z = 0 \) in an axisymmetric potential of the galaxy \( \Phi(r,z) \) (where \( r, z \) are cylindrical coordinates).

   (i) What is \( \Omega(r_0) \), the angular frequency of the star’s orbital motion?

   (ii) Suppose the star suffers a small perturbation so that its orbit varies as \( r(t) = r_0 + \Delta r(t) \) with \( |\Delta r| << r_0 \). Show that \( \Delta r \) exhibits harmonic oscillation and calculate the angular frequency \( \kappa \) for this oscillation (\( \kappa \) is called “epicyclic frequency”). Express \( \kappa \) in terms of \( \Omega(r) \) and its derivatives.

   (iii) How is \( \kappa \) related to the Oort constants? [see Eq.(2.63) of Schneider’s text.]

2. **(Gravitational Waves)** In class we explained that the power \( L \) of gravitational wave (GW) emission from an astrophysical source is given by (in order of magnitude)

   \[
   L \sim \frac{G}{c^5} \left| \frac{d^3Q}{dt^3} \right|^2.
   \]

   If the source is at a distance \( D \) from Earth, then the GW amplitude on Earth is

   \[
   h \sim \frac{G}{c^4D} \left| \frac{d^2Q}{dt^2} \right|.
   \]

   In the above, \( Q \) is the quadrupole moment of the source. Actually (as you may have seen from EM class), \( Q \) is a tensor, so really the above equations should be

   \[
   L \sim \frac{G}{c^5} \sum_{ij} \left| \frac{d^3Q_{ij}}{dt^3} \right|^2, \quad h_{ij} \sim \frac{G}{c^4D} \left| \frac{d^2Q_{ij}}{dt^2} \right|,
   \]

   where \( i, j \) stand for \( x, y, z \). For a binary system, we have, e.g., \( Q_{xx} = M_1 x_1^2 + M_2 x_2^2 \), where \( x_1 \) is the \( x \)-coordinate of \( M_1 \) relative to the center of mass, and similarly for \( x_2 \).

   Consider a binary consisting of two black holes (BHs) with \( M_1 = M_2 = 30 M_\odot \) in a circular orbit at a distance of 500 Mpc.

   (i) What is the separation \( a \) between the two BHs when the emitted GW frequency is 100 Hz. Calculate \( a/(GM_1/c^2) \), where \( M_t = M_1 + M_2 \). Note that as I mentioned in class, the GW wave frequency \( f \) is twice the orbital frequency. Use Newtonian mechanics in your calculation.

   (ii) Estimate the GW power \( L \) (in erg/s) when the binary is emitting GW at the frequency \( f = 100 \) Hz? Also estimate \( h \) at this frequency. If you are confused about the tensor notation, do not worry: For estimate, just use

   \[
   L \sim \frac{G}{c^5} \left| \frac{d^3Q_{xx}}{dt^3} \right|^2, \quad h \sim \frac{G}{c^4D} \left| \frac{d^2Q_{xx}}{dt^2} \right|.
   \]

   (iii) (Bonus): Show that, by measuring \( f \) and \( \dot{f} \) during binary inspiral, one can determine the “chirp mass” of the binary:

   \[
   M = \frac{(M_1 M_2)^{3/5}}{(M_1 + M_2)^{1/5}}.
   \]

   That is, one can express \( M \) in terms of \( f, \dot{f} \).