Astronomy 6570

Physics of the Planets
Secular Perturbations and Resonances
Secular Perturbations Between Satellites

Make approximations:
- \( m_{\text{sat}} \ll M_p \)
- \( e \ll 1 \), \( \sin I \ll 1 \)
- Neglect short-period perturbations (i.e., average over mean anomalies)

The disturbing function (potential of interaction) is given by, to \( 0(e^2) \) and \( 0(\sin^2 I) \):

\[
\mathbf{R}_i = n_i a_i^2 \left\{ \frac{1}{2} A_{ii} e_i^2 + \sum_{j \neq i} A_{ij} e_i e_j \cos(\varpi_i - \varpi_j) + \frac{1}{2} B_{ii} I_i^2 + \sum_{j \neq i} I_i I_j \cos(\Omega_i - \Omega_j) \right\}
\]

\[
A_{ii} = -B_{ii} = n_i \left\{ \frac{3}{2} J_2 \left( \frac{R}{a_i} \right)^2 + \frac{1}{4} \sum_{j \neq i} m_j F_1 \left( \frac{a_i}{a_j} \right) \right\}
\]

\[
A_{ij} = \frac{1}{4} n_i m_j F_2 \left( \frac{a_i}{a_j} \right)
\]

\[
B_{ij} = \frac{1}{4} n_i m_j F_1 \left( \frac{a_i}{a_j} \right)
\]

Note: \( A_{ii} = \) apsidal precession rate, neglecting variations in \( e_i \), and \( B_{ii} = \) nodal regression rate,

since \( \dot{\varpi} = \left( \frac{1}{na_i^2 e} \right) \frac{\delta R}{\delta e} \) and \( \dot{\Omega} = \left( \frac{1}{na_i^2 I} \right) \frac{\delta R}{\delta I} \)
Introduce new variables:

\[ h_i = e_i \sin \bar{\omega}_i \]
\[ k_i = e_i \cos \bar{\omega} \]
\[ p_i = \sin I_i \sin \Omega_i \]
\[ q_i = \sin I_i \cos \Omega_i \]
⇒ Differential equations:

\[
\begin{align*}
\frac{dh}{dt} &= \frac{1}{na^2} \frac{\partial \mathbb{R}}{\partial k} \\
\frac{dk}{dt} &= -\frac{1}{na^2} \frac{\partial \mathbb{R}}{\partial h} \\
\frac{dp}{dt} &= \frac{1}{na^2} \frac{\partial \mathbb{R}}{\partial q} \\
\frac{dq}{dt} &= -\frac{1}{na^2} \frac{\partial \mathbb{R}}{\partial p}
\end{align*}
\]

Substitute into $\mathbb{R}$, differentiate, and ...

\[
\begin{align*}
\dot{h}_i &= \sum_{j=i}^n A_{ij} k_j \\
\dot{k}_i &= -\sum A_{ij} h_j \\
\dot{p}_i &= \sum B_{ij} q_j \\
\dot{q}_i &= -\sum B_{ij} p_j
\end{align*}
\]
The solutions are of the form:

\[
\begin{align*}
  h_i &= e_{im} \sin(g_m t + \beta_m) \\
  k_i &= e_{im} \cos(g_m t + \beta_m)
\end{align*}
\]

where \(g_m\) and \(\{e_{im}, i = 1,2,\cdots,n\}\) are the eigenvalues and eigenvectors of the matrix \(A_{ij}\):

\[
\sum_{j=1}^{n} A_{ij} e_{jm} = g_m e_{im}
\]

i.e., \(A \cdot \mathbf{e}_m = g_m \mathbf{e}_m\)

Similarly for \((p_i,q_i)\) and the matrix \(B_{ij}\):

\[
B \cdot \mathbf{z}_m = g'_m \mathbf{z}_m
\]

The index "m" refers to the particular eigenmode \((1 \leq m \leq n)\).

Fit general solution to initial conditions \((e, \sigma, \sin I, \Omega)\) at some time by adjusting amplitudes, \(E_m\), and phases, \(\beta_m\), of the \(2n\) eigenmodes.

\[
E_m^2 = \sum_{i=1}^{n} e_{im}^2
\]
\[ R_j = n_j a_j^2 \left[ \frac{1}{2} A_{jj} e_j^2 + \sum_{k=1 \atop k \neq j}^n A_{jk} e_j e_k \cos(\tilde{\omega}_j - \tilde{\omega}_k) \\
\quad + \frac{1}{2} B_{jj} I_j^2 + \sum_{k=1 \atop k \neq j}^n B_{jk} I_j I_k \cos(\Omega_j - \Omega_k) \right] \]  \tag{1}

where

\[ A_{jj} = n_j \left[ \frac{3}{2} J_2 \left( \frac{R_p}{a_j} \right)^2 - \frac{9}{8} J_2 \left( \frac{R_p}{a_j} \right)^4 - \frac{15}{4} J_4 \left( \frac{R_p}{a_j} \right)^4 \\
\quad + \frac{1}{4} \sum_{k=1 \atop k \neq j}^n \frac{m_k}{M} \alpha_{jk} \bar{\alpha}_{jk} b_{3/2}^{(1)}(\alpha_{jk}) \right] \]  \tag{2}

\[ B_{ij} = -n_j \left[ \frac{3}{2} J_2 \left( \frac{R_p}{a_j} \right)^2 - \frac{27}{8} J_2 \left( \frac{R_p}{a_j} \right)^4 - \frac{15}{4} J_4 \left( \frac{R_p}{a_j} \right)^4 \\
\quad + \frac{1}{4} \sum_{k=1 \atop k \neq j}^n \frac{m_k}{M} \alpha_{jk} \bar{\alpha}_{jk} b_{3/2}^{(1)}(\alpha_{jk}) \right] \]  \tag{3}

\[ A_{jk} = -\frac{1}{4} \frac{m_k}{M} n_j \alpha_{jk} \bar{\alpha}_{jk} b_{3/2}^{(2)}(\alpha_{jk}) \]  \tag{4}

and

\[ B_{jk} = \frac{1}{4} \frac{m_k}{M} n_j \alpha_{jk} \bar{\alpha}_{jk} b_{3/2}^{(1)}(\alpha_{jk}) \]  \tag{5}
In these expressions

\[
\alpha_{jk} = \begin{cases} 
    a_k / a_j & \text{if } k < j \\
    a_j / a_k & \text{if } k > j
\end{cases}
\]  \quad (6)

\[
\bar{\alpha}_{jk} = \begin{cases} 
    1 & \text{if } k < j \\
    \alpha_{jk} & \text{if } k > j
\end{cases}
\]  \quad (7)

$n_j$ is the mean motion, $\bar{\omega}_j$ is the longitude of pericentre, $\Omega_j$ is the longitude of the ascending node, and $m_j$ is the mass, all for the $j$th satellite. $M$, $R$, $J_2$ and $J_4$, are, respectively, the mass, the radius and the first two zonal gravity coefficients of the planet, and the $b^{(k)}_s$ are Laplace coefficients$^{16}$. 
The solution of Lagrange's equations for the orbital element variations produced by this disturbing function follows the method described by Brouwer and Clemence. It is convenient to transform the conventional orbital elements into the symmetrical forms

\[ h = e \sin \omega \]

\[ k = e \cos \omega \]

and

\[ p = \sin I \sin \Omega \]

\[ q = \sin I \cos \Omega \]

in terms of which the solutions take the simple harmonic form:

\[ h_j = \sum_{i=1}^{n} e_{ji} \sin(g_i t + \beta_i) \]

\[ k_j = \sum_{i=1}^{n} e_{ji} \cos(g_i t + \beta_i) \]

and

\[ p_j = \sum_{i=1}^{n} l_{ji} \sin(f_i t + \gamma_i) \]

\[ q_j = \sum_{i=1}^{n} l_{ji} \cos(f_i t + \gamma_i) \]
The eccentricity eigenfrequencies, \( g_i \), are the roots of the determinant

\[
\begin{vmatrix}
A_{11} - g & A_{12} & \cdots & A_{1n} \\
A_{21} & A_{22} - g & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & \cdots & A_{nn} - g
\end{vmatrix} = 0
\] (12)

while the inclination eigenfrequencies, \( f_i \), are the roots of a similar determinant involving the elements \( B_{jk} \). The coefficients \( e_{ji} \) are components of the \( i \)th eigenvector of the matrix \( A \), with corresponding eigenvalue \( g_i \):

\[
\sum_{k=1}^{n} A_{jk} e_{ki} = g_i e_{ji}, \quad (j = 1, 2, \ldots, n)
\] (13)

Similarly, the coefficients \( l_{ji} \) \((j = 1, 2, \ldots, n)\) are components of the \( i \)th eigenvector of the matrix \( B \), corresponding to the eigenvalue \( f_i \). A complete solution of this form involves a total of \( 4n \) arbitrary constants—the magnitudes of the \( 2n \) eigenvectors, and the corresponding phases, \( \beta_i \) or \( \gamma_i \), which, in turn, are determined by a complete set of \( 4n \) initial conditions \((e_p, \omega_p, I_j, \Omega_j; j = 1, 2, \ldots, n)\) at some epoch. Note that the solutions for the eccentricity/pericentre variations are completely decoupled from the inclination/node solutions, thus considerably simplifying the analysis.
Geometric interpretations:

The secular solution for 2 planets has a very simple geometric interpretation in terms of \((h, k)\) coords:

for planet \(i\),

\[
\begin{align*}
    h_i &= e^+_i \sin \left( g^+_i t + \beta^+_i \right) + e^-_i \sin \left( g^-_i t + \beta^-_i \right) \\
    k_i &= e^+_i \cos \left( g^+_i t + \beta^+_i \right) + e^-_i \cos \left( g^-_i t + \beta^-_i \right)
\end{align*}
\]

- circle, radius \(e^+_i\)
- freq \(g^+_i\) (fast)
- circle, radius \(e^-_i\)
- freq \(g^-_i\) (slow)

It can be shown that \(\frac{e^+_2}{e^+_1} < 0\) and \(\frac{e^-_2}{e^-_1} > 0\) but the amplitudes \(\beta^\pm\) and phases are arbitrary:

* The overall value of \(e_i\) and \(\sigma_i\) are given by the vector sum of the two circulating components, rotating at angular frequencies \(g^+_i\) and \(g^-_i\).

* for \(n\) planets, the solution is similar, but with \(n\) circulating components rotating at \(g_1, g_2, \ldots, g_n\).
## Planetary Eigenfrequencies

### Eccentricities

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<th>Period (kyr)</th>
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<td>2</td>
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### Inclinations

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Mode 5 determines SS invariable plane

Brouwer & van Woerkom (1951)
## PLANETARY SECULAR EIGENFREQUENCIES

BVW (1950) values for $g_i$ & $f_i$:

<table>
<thead>
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<th>P (kyr)</th>
<th>Planets affected strongly</th>
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<td>$g_2 - g_4$</td>
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<td>$g_5 - g_6$</td>
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<td>N</td>
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<td>U, N (?)</td>
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<td>$s_8$</td>
<td>1913</td>
<td>U, N</td>
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</tbody>
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