Astronomy 6570

Physics of the Planets

Outer Planet Interiors
Giant Planets, Common Features

Mass: \(15 - 317 \, \text{M}_\oplus\)
Radius: \(3.9 - 11.2 \, \text{R}_\oplus\)
Density: \(0.69 - 1.67 \, \text{gcm}^{-3}\)
Rotation period: \(9.9 - 18\) hours
Obliq.: \(3° - 98°\)
Vis. Surf.: clouds; zonally banded (N?)
 decreasing contrast: J \(\rightarrow\) S \(\rightarrow\) U
Atmos. Comp.: \(\text{H}_2 + \text{He}\) (roughly solar)
 + \(\text{CH}_4, \text{NH}_3, \text{H}_2\text{O}, \ldots\) (enhanced by 3 – 5 in J)
Atmos. Struct.: adiabatic below \(\sim 1\) bar
 warm stratospheres
Energy output: \(\sim 2 \times\) solar input (exc. U)
Atmos. Circ'n.: Zonal winds of 100 – 400 ms
Mag. Field: 0.14 – 4.2 Gauss
 tilt = 0 – 59°
Satellites: inner, regular sats (e \(\sim\) i \(\sim\) 0)
 outer, irreg. sats
Ring system: increasing mass: J \(\rightarrow\) N \(\rightarrow\) U \(\rightarrow\) S
 assoc. with small satellites
 ephemeral structures?
Radio Occultation Temperature Profiles
Microwave emission originates from 0.5 – 10 bar levels

NH$_3$ absorption very strong near 1 cm \($\rightarrow$\) minimum in $T_B$.

• Condensation levels correspond to predicted cloud layers

• Only uppermost clouds observed directly
Planetary insolation patterns

- small obliquity (J)
- large obliquity (U)
Emitted infrared flux and equivalent brightness temperatures versus latitude for the four outer planets. The radiation is emitted, on average, from the 0.3 to 0.5 bar pressure levels. The equator-to-pole temperature differences are small. The largest temperature gradients occur at the extrema of the zonal velocity profile. (Ingersoll, 1990)
Zonal wind profiles (Voyager data)
Internal circulation models: Saturn

Fig. 12. Columnar convection cells (a) and cylindrical zonal flow (b). As shown by Busse (1976), the columnar mode is the preferred form of convective instability in a uniformly rotating, viscous, conducting fluid. The cylindrical mode is the most general form of steady zonal motion in an inviscid adiabatic fluid. The interaction of these two modes is analogous to the behavior of transverse convective disturbances in a sheared horizontal layer, according to Ingersoll and Pollard (1982).
Magnetic field comparison:

**Figure 2** Orientations of the planets and their magnetic fields.
## Comparison of planetary magnetic fields

<table>
<thead>
<tr>
<th></th>
<th>Earth (km)</th>
<th>Jupiter (km)</th>
<th>Saturn (km)</th>
<th>Uranus (km)</th>
<th>Neptune (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius, $R_{\text{planet}}$</td>
<td>6,373</td>
<td>71,398</td>
<td>60,330</td>
<td>25,559</td>
<td>24,764</td>
</tr>
<tr>
<td>Spin Period (Hours)</td>
<td>24</td>
<td>9.9</td>
<td>10.7</td>
<td>17.2</td>
<td>16.1</td>
</tr>
<tr>
<td>Magnetic Moment/M$_{\text{Earth}}$</td>
<td>1$^b$</td>
<td>20,000</td>
<td>600</td>
<td>50</td>
<td>25</td>
</tr>
<tr>
<td>Surface Magnetic Field (Gauss)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dipole Equator, $B_0$</td>
<td>0.31</td>
<td>4.2</td>
<td>0.22</td>
<td>0.23</td>
<td>0.14</td>
</tr>
<tr>
<td>Minimum</td>
<td>0.24</td>
<td>3.2</td>
<td>0.18</td>
<td>0.08</td>
<td>0.1</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.68</td>
<td>14.3</td>
<td>0.84</td>
<td>0.96</td>
<td>0.9</td>
</tr>
<tr>
<td>Dipole Tilt and Sense$^c$</td>
<td>+11.3°</td>
<td>-9.6°</td>
<td>-0.0°</td>
<td>-59°</td>
<td>-47°</td>
</tr>
<tr>
<td>Distance (A.U.)</td>
<td>1$^d$</td>
<td>5.2</td>
<td>9.5</td>
<td>19</td>
<td>30</td>
</tr>
<tr>
<td>Solar Wind Density (cm$^{-3}$)</td>
<td>10</td>
<td>0.4</td>
<td>0.1</td>
<td>0.03</td>
<td>0.005</td>
</tr>
<tr>
<td>$R_{\text{CF}}$</td>
<td>8 $R_E$</td>
<td>30 $R_J$</td>
<td>14 $R_S$</td>
<td>18 $R_U$</td>
<td>18 $R_N$</td>
</tr>
<tr>
<td>Size of Magnetosphere</td>
<td>11 $R_E$</td>
<td>50-100 $R_J$</td>
<td>16-22 $R_S$</td>
<td>18 $R_U$</td>
<td>23-26 $R_N$</td>
</tr>
</tbody>
</table>


$^b$ $M_{\text{Earth}} = 7.96 \times 10^{25}$ Gauss cm$^3 = 7.906 \times 10^{15}$ Tesla m$^3$.

$^c$ Note: Earth has a magnetic field of opposite polarity to those of the giant planets.

$^d$ 1 A.U. = 1.5 \times 10^8$ km.
Planetary Interior Models (general considerations)

Assume spherical symmetry (for simplicity only!)

1. Hydrostatic equilibrium: $\frac{dP}{dr} = -\rho g = \frac{-Gm(r)\rho}{r^2}$

2. Mass conservation: $\frac{dm}{dr} = 4\pi r^2 \rho$

3. Equation of state: $P = f(\rho, T; \text{composition})$

4. Heat transfer:
   \[ \begin{align*}
   k \frac{dT}{dr} &= F \cdots \text{conduction} \\
   \frac{dT}{dr} &= \left( \frac{\partial T}{\partial P} \right)_S \frac{dP}{dr} \cdots \text{convection}
   \end{align*} \]
   - in approximate treatments (4) may be dropped and (3) replaced by $P = f(\rho)$

3 first-order D.E.'s $\Rightarrow$ 3 boundary conditions

1. $m(0) = 0$

2. $P(R) = 0$

3. $T(R) = \begin{cases} T_{\text{surf}} \text{ (solid surface)} \\ T_{\text{eff}} \text{ (jovian planets)} \end{cases}$

Adjust model parameters (e.g., composition, $M_{\text{core}}$, etc.) to fit observables:

$M, R, J_2 \left( \sim \frac{c}{MR^2} \right), J_4, \text{ etc.}$

Model $\Rightarrow \rho(r), P(r), g(r), T(r)$
Central pressures:

\[ \frac{dP}{dr} = - \frac{Gm \rho}{r^2} \]

\[ \Rightarrow \text{roughly} \quad \frac{P_c - P_0}{R} \approx \frac{4\pi}{3} \frac{GM^2}{R^5} \]

\[ \therefore \quad P_c \approx \frac{GM^2}{R^4} \]

**Case I: a uniform-density planet:**

\[ m(r) = \frac{4\pi}{3} \rho r^3 \Rightarrow \]

\[ g = \frac{4\pi}{3} G \rho r \Rightarrow \]

\[ \frac{dP}{dr} = - \left( \frac{4\pi}{3} \right) G \rho^2 r \]

\[ \therefore \quad P(r) = P_c - \frac{2\pi}{3} G \rho^2 r^2 \]

Boundary condition \[ \Rightarrow P_c = \frac{2\pi}{3} G \rho^2 R^2 = \frac{3}{8\pi} \frac{GM^2}{R^4} \]
Case II:

\[ P = C \rho^2 \] (n = 1 polytrope \sim Jupiter)

\[ \Rightarrow P_c = \frac{2\pi^3}{9} G \rho^2 R^2 \] (see Polytrope notes)

Examples

<table>
<thead>
<tr>
<th>Body</th>
<th>( \rho \text{ (g cm}^{-3} )</th>
<th>( R \text{ (km)} )</th>
<th>Case I (( \text{Mb} = 10^{11} \text{Nm}^{-2} ))</th>
<th>Case II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moon</td>
<td>3.34</td>
<td>1738</td>
<td>0.047</td>
<td>0.155</td>
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<tr>
<td>Earth</td>
<td>5.52</td>
<td>6371</td>
<td>1.73</td>
<td>5.68</td>
</tr>
<tr>
<td>Uranus</td>
<td>1.32</td>
<td>25,600</td>
<td>1.60</td>
<td>5.25</td>
</tr>
<tr>
<td>Jupiter</td>
<td>1.33</td>
<td>71,400</td>
<td>12.6</td>
<td>41.4</td>
</tr>
</tbody>
</table>

Detailed models: Earth = 3.5 Mb
                Jupiter = 70 Mb
                Uranus = 8 Mb
Adiabatic equations of state (EOS) for hydrogen, "ice", and "rock".
Zero-temperature planet models for pure components.
Hydrogen phase diagram.
Giant Planet Models (Stephenson)
Polytropes

Interior structure equations:

\[
\frac{dm}{dr} = 4\pi r^2 \rho \quad (1)
\]
\[
\frac{dP}{dr} = -g\rho = -\frac{Gm\rho}{r^2} \quad (2)
\]
\[
\rightarrow m(r) = -\frac{r^2}{G\rho} \frac{dP}{dr}
\]
\[
\therefore \quad \frac{dm}{dr} = -\frac{1}{G} \frac{d}{dr}\left(\frac{r^2}{\rho} \frac{dP}{dr}\right) = 4\pi r^2 \rho \quad \text{... from (1)}
\]

We assume \( P = \mathbb{C} \rho^{\frac{1}{n+1}} \quad \mathbb{C} = \text{constant} \quad (3) \)

and write \( \rho = \rho_0 \theta^n \Rightarrow P = P_0 \theta^{n+1} \)

with \( P_0 = \mathbb{C} \rho_0^{\frac{1}{n+1}} \)

Substitute in the DE:

\[
\frac{d}{dr}\left(r^2 \frac{d\theta}{dr}\right) + \left(\frac{4\pi G \rho_0^2}{(n+1)P_0}\right) r^2 \theta^n = 0 \quad (4)
\]

\[
\Rightarrow \quad \text{set} \quad \xi = \frac{r}{r_0}, \quad \text{with} \quad r_0 \equiv \left(\frac{(n+1)P_0}{4\pi G \rho_0^2}\right)^{\frac{1}{2}}
\]
\[
\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \xi^2 \theta^n = 0 \quad \text{... the LANE-EMDEN equation.} \quad (5)
\]
Boundary conditions:

\[ \rho = \rho_0 \text{ at } r = 0 \Rightarrow \theta(0) = 1 \]  \hspace{1cm} (a)

\[ P = 0 \text{ at } r = R \Rightarrow \theta(\xi_1) = 0 \]  \hspace{1cm} (b)

where

\[ \xi_1 = \frac{R}{r_0}. \]

Also \( \rho \geq 0 \) at all \( 0 \leq r < R \Rightarrow \xi_1 = 1^{st} \text{ zero of } \theta(\xi). \)

Also we require \( \frac{dP}{dr} \propto \theta^n \frac{d\theta}{d\xi} = -g\rho \rightarrow 0 \text{ as } r \rightarrow 0 \)

\[ \therefore \frac{d\theta}{d\xi}(0) = 0 \]  \hspace{1cm} (c)

So starting with (a) & (c), the L-E equation is integrated outwards until \( \theta = 0 \) for the first time.
Polytropes (cont’d.)

**Free parameters:**

\[ \{n, \rho_0, r_0\} \Rightarrow P_0, \mathbb{C} \ - \text{fit to observed } M, R, J_2 \]

\[ R = r_0 \xi_1 \]

\[ M = m(R) = - \frac{R^2}{G} \left( \frac{1}{\rho} \frac{dP}{dr} \right)_R \]

\[ = - \frac{r_0^2 \xi_0^2}{G} \cdot \frac{(n+1)P_0}{\rho_0} \left. \frac{d\theta}{d\xi} \right|_{\xi_1} \]

i.e.,

\[ M = -4\pi \rho_0 r_0^3 \xi_1^2 \left. \frac{d\theta}{d\xi} \right|_{\xi_1} \]

and

\[ \frac{\bar{\rho}}{\rho_0} = 3 \left. \frac{d\theta}{d\xi} \right|_{\xi_1} \]

so for a given value of \(n,(M,R) \rightarrow r_0, \rho_0\).
Polytropes (cont’ d.)

\( J_2 \) is determined by the moment of inertia, \( \frac{C}{MR^2} \). For a spherical planet

\[ A = B = C \text{ and } A + B + C = 3C = 2 \int r^2 \, dm \]

\[ :. \quad C = \frac{2}{3} \int_0^R r^2 \, dm = \frac{8\pi}{3} \int_0^R r^4 \, \rho(r) \, dr \]
\[ = \frac{8\pi}{3} \rho_0 \int_0^{\xi_0} \xi^4 \theta^n \, d\xi \]

Now

\[ \Pi = -\int \xi^2 \frac{d}{d\xi} \left( \xi^2 \theta' \right) \, d\xi \]
\[ = - \xi_1^4 \theta'_1 + 2 \int \xi_1^3 \theta' \, d\xi \]
\[ = - \xi_1^4 \theta'_1 + 2 \xi_1^3 \theta'_1 - 6 \int \xi_1^2 \theta \, d\xi \]
\[ = - \xi_1^4 \frac{d\theta}{d\xi} \Bigg|_{\xi_1} - 6 \int_0^{\xi_1} \xi^2 \theta \, d\xi \]

and

\[ \frac{C}{MR^2} = \frac{-2\Pi}{3\xi_1^4 \theta'} \]
Summary of Properties of Polytropes:

\[ R = \xi_1 r_0 \]

\[ \bar{\rho} = \frac{3}{\xi_1} \left( \frac{d\theta}{d\xi} \right)_1 \rho_0 \]

\[ P_0 = \frac{4\pi G \rho_0^2 r_0^2}{n+1} = \alpha \frac{GM^2}{R^4} \]

\[ P.E. = -\beta \frac{GM^2}{R} \]

\[ M = -4\pi \rho_0 r_0^3 \xi^2_1 \left( \frac{d\theta}{d\xi} \right)_1 \]
Polytropic Models of Jovian Planets:

\[ \frac{J_2}{q} = \frac{1}{3} k_2 \]

\[ q = \frac{\omega^2 R^3}{GM} \]

Conclusions:

\( n \approx 0.95 \) \( \ldots \) Jupiter
1.3 \( \ldots \) Saturn
1.5 \( \ldots \) Uranus
1.2 \( \ldots \) Neptune

Note: For \( n = 1 \), 
\[ r = \left( \frac{C}{2\pi G} \right)^{\frac{1}{2}} \]

independent of \( \rho_0 \) and \( M \)
Figure 1: Density profiles in models of Jupiter (gray line) and Saturn (continuous lines: adiabatic i-EOS and PPT-EOS models; dashed: non-adiabatic i-EOS model). Upper curves (dashed and dot-dashed) are $T = 0$ K density profiles for water ice and olivine (from Thompson 1990). The dashed region represents the assumed uncertainty on the EOS for heavy elements ($\rho_Z(P,T)$). Within this region, the continuous line corresponds to our “preferred” profile for $\rho_Z$. Inset: Differences of the decimal logarithm of the Saturn density profiles with the same profile using the i-EOS and an adiabatic structure (plain and dotted lines). The gray line corresponds to the same difference but for a PPT-EOS non-adiabatic Jupiter model. (Adapted from Guillot et al. 1997).
Figure 3: Optimization of a model of Jupiter. The surface represents the decimal logarithm of the minimization function $\chi^2$ (see Eq. 9), as a function of the core mass $M_{\text{core}}$ and the hydrogen mass mixing ratio $X = 1 - Y_Z$ in the molecular and metallic regions ($\Delta Y_Z = 0$). Each point of the $\chi^2$ surface represents a converged interior model.
Figure 6: Constraints Jupiter's (upper panel) and Saturn's (lower panel) core masses ($M_{\text{core}}$) and total masses of heavy elements ($M_{\text{tot}}$), expressed in Earth masses ($M_\oplus$). The dashed regions correspond to the constraints obtained from static models only. The shaded regions are those which also satisfy the condition that the model ages should be close to the age of the Solar System (see text). Within those, darker regions correspond to calculations that ignore uncertainties on the equation of state of heavy elements. Models calculated with the PPT-EOS are to the left, models using the i-EOS to the right. The region surrounded by the thick line corresponds to the most plausible ($M_{\text{core}}, M_{\text{tot}}$) region given all possible constraints and uncertainties. In the case of Saturn, the horizontally-dashed region and the region surrounded by dashed lines correspond to constraints using the Voyager helium mixing ratios $Y = 0.06 \pm 0.05$, whereas other models assume $Y = 0.16 \pm 0.05$. Arrows indicate the direction and magnitude of the assumed uncertainties, if $J_4$ or $Y_{\text{proto}}$ are increased by 1σ, rotation is assumed to be solid ("Ω"), the core is assumed to be composed of ices only ("$f_{\text{ice}}$"), if Jupiter's interior becomes fully adiabatic ("$\nabla_T$"), and if Saturn's surface temperature is increased from 135 to 145 K ("$T_{\text{bar}}$").
Figure 7: Same as Fig. 6 but for constraints on the mass fractions of heavy elements in the molecular envelope \( (Z_{\text{mol}}) \), and in the metallic envelope \( (Z_{\text{met}}) \). The \( Z_{\text{mol}} = Z_{\text{met}} \) relations are shown by diagonal dotted lines. Additional arrows not included in Fig. 6 show the changes due to a 1-\( \sigma \) increase in the surface helium mass mixing ratio \( (Y^{\text{mol}}) \), and in the case of Jupiter, of an increase in surface temperature from 165 to 170K \( (T_{1\text{bar}}) \). The mixing ratios are in solar units \( (Z_{\odot} = 0.0192; \text{see Anders \\& Grevesse 1992}) \). \textbf{Inset:} closeup view of the solutions and assumed uncertainties for Jupiter models calculated with the PPT-EOS.
Figure 9: Gravitational moments $J_4$ and $J_6$ of the optimized models of Jupiter (right) and Saturn (left). Large open symbols represent models that satisfy all constraints. Small filled symbols represent models that match the observed gravitational field but do not predict ages in agreement with that of the Solar System. Squares are models with no core, all other models being represented by circles. The plain crosses are the observational constraints on $(J_4, J_6)$. The dashed crosses show the constraints derived when differential rotation is assumed to reach the deep interior. Inset is an enlargement of the solutions obtained for Jupiter.
Podolak et al. (1989) Uranus
Podolak et al. (1989) Uranus

Models fit Current J₂ & J₄
Density distribution in Uranus, calculated for a model with solar abundances of “ice” and “rock”. The temperature distribution corresponds to the present epoch.
Density distribution in Neptune, calculated for a model with solar abundances of “ice” and “rock”. The temperature distribution corresponds to the present epoch.
Notes on Giant Planet Interiors:

• Internal heat sources in J, S & N probably due to continued gravitational energy release thru contraction: ~ 30 km/My for Jupiter.

• Models predict that Jupiter is also cooling at ~ K/My.

• Lack of internal heat source in Uranus may be a consequence of subtle compositional gradients inhibiting interior convection. Observations can be matched if there is no convection interior to ~ 0.6 $R_u$ → magnetic field must be generated in the outer icy mantle.

• D/H rations in U and N are ~ $10^{-4}$, similar to Earth and comets but ~ 10 × larger than in J and S → composition dominated by ices rather than $H_2 +$ He.