Momenergy

Relativity and Astrophysics
Lecture 25
Terry Herter

Outline

- Momenergy
  - Momentum-energy 4-vector
  - Magnitude & components
  - Invariance
  - Low velocity limit
- Concept Summary

- Reading
  - Spacetime Physics: Chapter 7
- Homework: (due Wed. 11/04/09)
  - 6-4, 6-7, 7.3, and 7.9
The Power of It All

- The conservation law all us to solve for quantities without knowing the details:
  - We don’t have to know how the objects deform in the sticking case
  - We don’t need to know about the details for the collisions at all (for the completely inelastic and elastic cases)

- In cases where this a potential we can include this in the energy conservation equations:
  - For instance, using Newton’s law of gravity we can write a conservation of energy equation which relates velocity to distance from the Earth – we don’t have to solve the details of the acceleration
  - For a vertically falling object we have:
    \[ F = -\frac{GMm}{r^2} \Rightarrow v = \frac{GMm}{r} \Rightarrow \frac{1}{2}mv^2 = GMm\left(\frac{1}{r_o} - \frac{1}{r}\right) \]
  - Where the object starts at \( r_o \) with zero velocity.
  - Note that we lose information. We don’t know how long it takes to travel over this distance – just the speed at the end.

Momenergy

- Momenergy
  - Relativity combines momentum and energy into a single concept, momentum-energy (or momenergy)
  - This quantity is conserved in a collision

- Momenergy proportional to mass
  - Consider different mass pebbles hitting a windshield

- Momenergy is a directed quantity
  - It matters which direction the pebble come from
  - Momenergy is a 4-vector
    - Expect space and time components due to the unity of spacetime (three spatial parts and one time part)
      - Space part represent momentum, time part represents energy
    - Points in the direction of a particles spacetime displacement

- Momenergy is reckoned using proper time for a particle
  - Momenergy is independent of reference frame

  \[
  \text{momenergy} = (\text{mass}) \frac{(\text{spacetime displacement})}{(\text{proper time for that displacement})}
  \]

- Looks like Newtonian momentum but modified for Einstein’s relativity
Magnitude of Momenergy

- Don’t confuse a 4-vector with its magnitude
  - The proper time is the magnitude of the spacetime displacement
  - The fraction is a unit 4-vector pointing in the direction of the worldline of the particle
  - The magnitude of momenergy is its mass.

Components of Momenergy

- Let $\tau$ stand for proper time then the components of momenergy are
  
  
  \[
  E = m \frac{dt}{d\tau}, \quad p_i = m \frac{dx_i}{d\tau}
  \]

- Let’s look at its magnitude
  
  \[
  (\text{magnitude})^2 = E^2 - p_i^2 = m^2 \quad (i = \text{space})
  \]

- Or more compactly written
  
  \[
  (\text{magnitude of momenergy arrow})^2 = E^2 - p^2 = m^2
  \]

- which is just the equation for a hyperbola in spacetime again.
- At right is a plot of the momenergy 4-vector for a single particle observed in 5 different inertial reference frames.
Momentum: “Space Part”

- Consider a particle moving along the $x$-axis with a velocity $v$ in the lab frame.
- The displacement of the particle is $x = vt$, or for small displacements, $dx = v \, dt$.
- The proper time is:
  
  $$\tau = \left( \frac{dt^2 - dx^2}{c^2} \right)^{1/2} = \left( dt^2 - v^2 \, dt \right)^{1/2}$$

- So that the relativistic expressions for energy and momentum are
  
  $$E = m \frac{dt}{d\tau} = m \gamma \quad \& \quad p = m \frac{dx}{d\tau} = m \frac{dx}{dt} \frac{dt}{d\tau} = mv\gamma$$

- For low velocities the momentum expression becomes very close to the Newtonian value.

Momentum Units

- Relating velocities (dimensionless vs. conventional units)
  
  $$v = \frac{v_{\text{conv}}}{c}$$

- For a momentum in dimensionless units
  
  $$p_{\text{Newton}} = mv \quad \text{Valid for low speed}$$
  
  $$p = mv\gamma \quad \text{Valid at any speed}$$

- For momentum in conventional units
  
  $$p_{\text{conv Newton}} = p_{\text{Newton}} c = mvc = mv_{\text{conv}} \quad \text{Valid for low speed}$$
  
  $$p_{\text{conv}} = pc = mvc\gamma = mv_{\text{conv}}\gamma \quad \text{Valid at any speed}$$

- Convert from momentum in units of mass to conventional units by multiplying by $c$, the speed of light.
Energy: “Time Part”

- For a particle moving along the x-axis with a velocity \( v \) in the lab frame the energy is
  \[
  E = m \frac{dt}{d\tau} = m\gamma
  \]
- Which can be compared with the Newtonian expression (using \( K \) as the symbol for kinetic energy)
  \[
  K = \frac{1}{2}mv^2
  \]
- How does the relativistic expression for energy compare with the Newtonian for kinetic energy?
- At low velocities, \( v \approx 0 \), we have
  \[
  E_{\text{rest}} = m
  \]
- Which is called the rest energy of the particle.
  - Rest energy is simply the mass
  - The relativistic energy does not go to zero like the \( K \)
- So to define a kinetic energy above and beyond a particle’s rest energy we have
  \[
  K = E - E_{\text{rest}} = E - m = m(\gamma - 1)
  \]

Energy Units

- Note that if we divide momentum and energy we get the speed of the particle
  \[
  E = m \frac{dt}{d\tau} = m\gamma \quad \& \quad p = mv \gamma \quad \Rightarrow v = \frac{p}{E}
  \]
- To convert energy in units of mass to energy in conventional units we have
  \[
  E_{\text{conv}} = E c^2 = mc^2 \gamma
  \]
- The rest energy (and perhaps the most famous equation in physics) is
  \[
  E_{\text{rest}} = mc^2
  \]
- Particle at rest
- The kinetic energy is
  \[
  K_{\text{rest}} = (E - E_{\text{rest}}) c^2 = mc^2 (\gamma - 1)
  \]
- Valid at any speed
- \( \gamma = \left(1 - v^2\right)^{-1/2} \)
- At low speeds (\( v \ll 1 \)), we have
  \[
  \gamma - 1 \approx \left(1 + \frac{1}{2}v^2\right) - 1 = \frac{1}{2}v^2
  \]
  \[
  K_{\text{conv Newton}} = \frac{1}{2}mv^2 c^2 = \frac{1}{2}mv^2 \gamma_{\text{conv Newton}}
  \]
- Valid at low speed
Sample Problem 7-2 (pg. 202)

- Consider a 3 kg mass object which moves 8 meters in the x direction in 10 meters of time.
  - What is its energy and momentum?
  - What is its rest energy?
  - What is its kinetic energy? Compare this to the Newtonian KE.
  - Verify the velocity equals its momentum divided by its energy.

- The speed is
  \[ v = \frac{x}{t} = \frac{8 \text{ m}}{10 \text{ m}} = 0.8 \]
  \[ \gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{1}{\sqrt{1 - 0.8^2}} = \frac{1}{0.36} = \frac{1}{0.6} = \frac{5}{3} \]

- The energy and momentum are
  \[ E = m\gamma = 3 \text{ kg} \times \frac{5}{3} = 5 \text{ kg} \]
  \[ p = mv\gamma = (3 \text{ kg}) \times 0.8 \times \frac{5}{3} = 4 \text{ kg} \]

- The rest energy is
  \[ E_{\text{rest}} = m = 3 \text{ kg} \]

- The relativistic and Newtonian kinetic energies are
  \[ K = E - E_{\text{rest}} = 2 \text{ kg} \]
  \[ K_{\text{Newton}} = 0.5 \text{ m}^2 s^{-2} \times 3 \text{ kg} \times (0.8)^2 = 0.96 \text{ kg} \]

- The correct relativistic result is quite a bit larger than the Newtonian prediction, and in fact the correct result grows with out limit as \( v \) approaches 1 (more on next slide).

- Finally the velocity can be recovered from
  \[ v = \frac{p}{E} = \frac{4}{5} = 0.8 \]

Energy in the low-velocity limit

- In terms of momentum the expression for energy looks like
  \[ E^2 = m^2 + p^2 \Rightarrow E = \sqrt{m^2 + p^2} \]

- At low velocities \( p/m \) is small, and we can use the usual expansion to get
  \[ (1 + x)^2 = 1 + nx + \text{corrections} \Rightarrow \frac{E}{m} = 1 + \frac{1}{2} \frac{p^2}{m^2} + \text{corrections} \]

- Suppose \( (p/m)^2 = 0.21 \), then the approximate and exact formula give respectively
  \[ \text{approximate} \quad \frac{E}{m} = 1 + \frac{1}{2} \left( \frac{p}{m} \right)^2 = 1.105 \]
  \[ \text{exact} \quad \frac{E}{m} = \left[ 1 + \frac{p^2}{m^2} \right]^{1/2} = (1.21)^{1/2} = 1.100 \]

- The correction is negative and very small: correction = -0.005
Low-velocity limit (cont’d)

- In terms of velocity the expression for energy looks like
  \[ E = m\gamma = \frac{m}{\sqrt{1 - v^2}} \Rightarrow \frac{E}{m} = \frac{1}{\sqrt{1 - v^2}} \]

- At low velocities \( v \) is small, and we can use the usual expansion to get
  \[ (1 + x)^n = 1 + nx + \text{corrections} \Rightarrow \frac{E}{m} = 1 + \frac{1}{2} v^2 + \text{corrections} \]

- Suppose \( v^2 = 0.19 \), then the approximate and exact formula give respectively
  
  \[
  \begin{align*}
  \text{approximate} & \quad \frac{E}{m} = 1 + \left(\frac{1}{2} v^2\right) = 1 + 0.19 = 1.095 \\
  \text{exact} & \quad \frac{E}{m} = \left(\frac{1}{\sqrt{1 - v^2}}\right)^{1/2} = \left(\frac{1}{0.9}\right)^{1/2} = 1.111... \\
  \end{align*}
  \]

- The correction is positive and very small: correction = +0.01611

KE: Correct vs. Newtonian

- The correct (relativistic) result for the kinetic energy is
  \[ K = E - E_{\text{rest}} = E - m = m(\gamma - 1) \]

  - The correct \( K \) increases without bound as \( v \) approaches 1 while the Newtonian result approaches 0.5 (for a 1 kg mass)

- Plots:
  - The left hand plot shows the correct vs. the low velocity Newtonian approximation (which extrapolates incorrectly to high velocities).
  - The right hand plot shows the percentage error in the Newtonian result compared to the correct one.