Momentum and Energy

Relativity and Astrophysics
Lecture 24
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Outline

- Newtonian Physics
  - Energy and Momentum
  - Conservation of energy and momentum

- Reading
  - Spacetime Physics: Chapter 7
- Homework: (due Wed. 11/04/09)
  - 6-4 and 6-7 (maybe more on Friday)
Sample Problem 6-2 (pg. 178)

- Astronaut shouts out "Damn!" at 12:00 GMT but after one second (after 12:00 GMT) a short circuit temporarily disables receiver at Mission Control on Earth. Take \(d = 3.84 \times 10^8\) m between Earth and Moon in Earth frame.
- Does mission control here expletive?
  - No, since \(c = 3 \times 10^8\) m/sec. Not enough time to travel to Earth before short occurs.
- Could astronauts language have cause the short?
  - No, since signal can’t travel faster than light.
- How do we classify the spacetime separation?
  - Spacelike – space separation > time separation.
- What is proper distance?
  - \(s^2 = (3.84 \times 10^8)^2 - (3 \times 10^8)^2 \Rightarrow s = 2.4 \times 10^8\) m
- What is shortest distance?
  - The shortest distance is equal to the proper distance.

Newtonian Momentum & Energy

- Conservation of Momentum
  - In the absence of a force we have
    \[ F = ma \Rightarrow m \frac{dv}{dt} = 0 \Rightarrow mv = \text{constant} = p \]
  - So that the momentum \((p)\) is conserved.
- Conservation of Energy
  - Writing the force, \(F\), as the derivative of a potential, \(V\), we have
    \[ F = -\frac{dV}{dr} = m \frac{dv}{dt} \Rightarrow -dr \frac{dV}{dr} = m \frac{dv}{dt} dr \Rightarrow -dV = mvdv \Rightarrow \frac{1}{2} mv^2 + V = E \]
  - The energy, \(E\), of the system is conserved. Note: \(KE = \frac{1}{2} mv^2 = \frac{p^2}{2m}\).
- Conservation laws are very general in physics. Symmetries result in a conserved quantity.
  - Shift symmetry in time => conservation of energy
  - Shift symmetry in space => conservation of momentum
  - For further info a place to start is Wikipedia article on “conservation of energy”
- The use of conservation laws can greatly simplify the solution of otherwise complex physical problems.
Conservation of Momentum Example

Before Collision

\( m_1 = 1 \text{ ton} \quad v_1 = 69 \text{ mph} \)

\( m_2 = 2 \text{ ton} \quad v_2 = -60 \text{ mph} \)

After Collision

\( m' = 3 \text{ ton} \quad v' = ? \)

- Suppose we two moving objects collide and stick. What is their final velocity?
- Since momentum is conserved we have

\[
\begin{align*}
  m_1 v_1 + m_2 v_2 &= p = p' = m' v' \\
  v' &= \frac{m_1 v_1 + m_2 v_2}{m'}
\end{align*}
\]

- So that the final velocity is -17 mph.
  - The final, joined mass is moving to the left.

- Note – energy is conserved but some of it went into deformation of the object
  - And possibly heat, sound and light which could carry away momentum, so we assumed this is a small effect.

- We now look at the case where they bounce off one another.

Conservation of Energy Example

Before Collision

\( m_1 = 1 \text{ ton} \quad v_1 = 69 \text{ mph} \)

\( m_2 = 2 \text{ ton} \quad v_2 = -60 \text{ mph} \)

After Collision

\( v_1' = ? \quad v_2' = ? \)

- Suppose we two moving objects collide elastically (negligible deformation, heating, sounds, etc). What are their final velocities?
- We use both momentum and energy conservation

\[
\begin{align*}
  m_1 v_1 + m_2 v_2 &= p = p' = m' v' \\
  \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 &= \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2
\end{align*}
\]

- There are two equations with two unknowns.
  - This can be solved by "brute force" but let's try a somewhat more elegant approach

- Define the center of mass (CM) which tracks the net momentum of the system \( p_{\text{cm}} = p_1 + p_2 \) which is conserved before and after the collision

\[
p_{\text{cm}} = (m_1 + m_2) v_{\text{cm}} = p_1 + p_2 = m_1 v_1 + m_2 v_2
\]

- which defines \( v_{\text{cm}} \) and the velocities in CM coordinate system

\[
\begin{align*}
  v_{1,\text{cm}} &= \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \\
  v_{2,\text{cm}} &= v_2 - v_{1,\text{cm}}
\end{align*}
\]
Energy Example (cont’d)

Before Collision

\[ m_1 = 1 \text{ ton} \]
\[ v_1 = 69 \text{ mph} \]

After Collision

\[ m_2 = 2 \text{ ton} \]
\[ v_2 = -60 \text{ mph} \]

- Now define momentum of particle 1 relative to CM

\[ p_{1,cm} = m_1(v_1 - v_{cm}) = \frac{m_1 m_2}{m_1 + m_2} (v_1 - v_2) \approx \mu (v_1 - v_2) \]

- Likewise for particle 2

\[ p_{2,cm} = \mu (v_2 - v_1) = -p_{1,cm} \]

- So that

\[ p_{1,cm} + p_{2,cm} = 0 \]

The conservation equations can now be written as

\[ p_{1,cm}' + p_{2,cm}' = \frac{p_{1,cm}^2}{2m_1} + \frac{p_{2,cm}^2}{2m_2} \]

The solution to these equations gives (see right)

\[ p_{1,cm}' = \pm p_{1,cm} \quad \text{&} \quad p_{2,cm}' = \pm p_{2,cm} \]

The positive solution is the case where they miss

\[ v_{1}' = \pm v_{1} \quad \text{&} \quad v_{2}' = \pm v_{2} \]

Energy Example (cont’d)

Before Collision

\[ m_1 = 1 \text{ ton} \]
\[ v_1 = 69 \text{ mph} \]

After Collision

\[ m_2 = 2 \text{ ton} \]
\[ v_2 = -60 \text{ mph} \]

- We can look at the change in momentum (impulse)

\[ \Delta p_{1,cm} = p_{1,cm}' - p_{1,cm} = -2 p_{1,cm} \quad \Rightarrow \quad \Delta p_{2,cm} = -\Delta p_{1,cm} \]

We can see from definition of \( v_{1,cm} \) and \( v_{2,cm} \) that

\[ \Delta v_1 = \Delta p_{1,cm} \quad \text{&} \quad \Delta v_2 = \Delta p_{2,cm} \]

So that

\[ \Delta p_{1,cm} = \frac{2m_1 m_2}{m_1 + m_2} (v_2 - v_1) \quad \Rightarrow \quad \Delta v_1 = \frac{\Delta p_{1,cm}}{m_1 + m_2} (v_2 - v_1) \]

And likewise for \( \Delta v_2 \) (switch 1 and 2 in the above two equations)
Energy Example (cont’d)

Before Collision

\[ m_1 = 1 \text{ ton} \quad v_1 = 69 \text{ mph} \]

\[ m_2 = 2 \text{ ton} \quad v_2 = -60 \text{ mph} \]

Thus we get

\[ v_1' = v_1 + \Delta v_1 = v_1 + \frac{2m_2}{m_1 + m_2} (v_2 - v_1) \]

\[ v_2' = v_2 + \Delta v_2 = v_2 + \frac{2m_1}{m_1 + m_2} (v_1 - v_2) \]

In the limiting case when \( m_1 = m_2 \) we see that

\[ v_1' = v_2 \quad \& \quad v_2' = v_1 \]

So that the particles switch velocity when they have the same mass

- Such as in pool, the cue ball stops and the impacted ball flies off

Note – we can use vectors for the velocity and momentum and this all works in 2 and 3 dimensions

We can determine the final velocities for our example

\[ v_1' = v_1 + \frac{2m_2}{m_1 + m_2} (v_2 - v_1) = \frac{2 \cdot 60}{1 + 2} (69 - 60) = -103 \text{ mph} \]

\[ v_2' = v_2 + \frac{2m_1}{m_1 + m_2} (v_1 - v_2) = -60 + \frac{2 \cdot 1}{1 + 2} (69 + 60) = 26 \text{ mph} \]

Which shows us that the light object is rebounded at a higher speed (in this case) than it had originally

If we consider the case when \( m_2 \gg m_1 \) then we have

\[ v_1' = v_1 + \frac{2m_2}{m_1 + m_2} (v_2 - v_1) \]

\[ v_2' = v_2 + \frac{2m_1}{m_1 + m_2} (v_1 - v_2) \]

Energy Example (cont’d)
The Power of It All

- The conservation law all us to solve for quantities without knowing the details
  - We don’t have to know how the objects deform in the sticking case
  - We don’t need to know about the details for the collisions at all (for the completely inelastic and elastic cases)

- In cases where this a potential we can include this in the energy conservation equations.
  - For instance, using Newton’s law of gravity we can write a conservation of energy equation which relates velocity to distance from the Earth – we don’t have to solve the details of the acceleration
  - For a vertically falling object we have

\[ F = \frac{GMm}{r^2} \Rightarrow V = \frac{GMm}{r} \Rightarrow \frac{1}{2}mv^2 = GMm \left( \frac{1}{r_o} - \frac{1}{r} \right) \]

- Where the object starts at \( r_o \) with zero velocity.
- Note that we lose information. We don’t know how long it takes to travel over this distance – just the speed at the end.