The Lorentz Contraction

Relativity and Astrophysics
Lecture 07
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Outline

- Proof of contraction along the direction of propagation
- Worked Problems
  - 3-14a,c and 3-15
- No faster than light travel
  - Box L-1 (page 108-9) of textbook

- Reading
  - Special Topic: Lorentz Transformation
- Homework: (Due Wed, 9/16/09)
  - 3-1, 3-7, 3-10, 3-16
Velocity addition: another example

- Person C sees A and B moving as shown above.
- How fast does A think B is moving?
  - Choose A as "lab" frame, C in rocket frame (moving with \( v_{\text{rel}} \)), and B as the bullet (with velocity \( v' \) in rocket frame).
  - So we are asking for velocity of B (bullet) in lab frame.

\[
v = \frac{v' + v_{\text{rel}}}{1 + v'v_{\text{rel}}} = \frac{-0.95 + 0.9}{1 - 0.95 \times 0.9} = \frac{-0.05}{1 - 0.855} = -0.345
\]

Length Contraction - Derivation

- Consider two photon clocks (1 and 2 below) oriented perpendicular to one another on the train.
- To an observer on the train \( d_T = d_L \).
- What about an outside observer?
We already know about clock (1). For a double pass between the mirrors an outside observer the time is:

\[ t = \frac{t'}{\sqrt{1-v_{rel}^2}} = \frac{2d_f}{\sqrt{1-v_{rel}^2}} \]

- The time, \( t' \), here is the time to make a full circuit.
- Note that time here is in units of length.

For clock (2), the mirrors are constantly moving to the right. Three key snapshots are:

- Photon leaves
- Photon reflects
- Photon arrives back at first mirror
Let $t_o$ be the time for the photon to go from the left mirror to the right mirror.

- The right mirror then travels away a distance $vt_o$ before the photon hits it.
- Hence, the photon travels a distance $d_o$ given by:

$$d_o = v_{rel}t_o + d_L$$

but $d_o = t_o$ \implies $t_o = \frac{d_L}{1 - v_{rel}}$

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Let $t_1$ be the time for the photon to go from the right mirror to the left mirror.

- The left mirror then travels a distance $vt_1$ towards the photon before it hits.
- Hence, the photon travels a distance $d_1$ given by:

$$d_1 = d_L - v_{rel}t_1$$

but $d_1 = t_1$ \implies $t_1 = \frac{d_L}{1 + v_{rel}}$
The total time for a round trip is:

\[ t = t_o + t_1 = \frac{d_L}{1-v_{rel}} + \frac{d_L}{1+v_{rel}} \]

Combining terms

\[ t = \frac{2d_L}{1-v_{rel}^2} \]

But this must give the same time as clock (1).

\[ t = \frac{2d_T}{\sqrt{1-v_{rel}^2}} \]

Making the time for clocks (1) and (2) equal

\[ \frac{2d_T}{\sqrt{1-v_{rel}^2}} = \frac{2d_L}{1-v_{rel}^2} \]

So that we must have

\[ d_L = d_T \sqrt{1-v_{rel}^2} \]

The longitudinal distance is smaller than the transverse distance.

Thus, in the direction of motion, objects contract!
Lorentz Contraction

Time Dilation

\[ t' = t \left(1 - \frac{v^2}{c^2}\right)^{1/2} \]

- Curve shows “aging” of moving object
  - That is, time passes more slowly
  - At \( v = 1 \), the clock appears to stop !!!

Length Contraction

\[ d_L = d_T \sqrt{1 - \frac{v^2}{c^2}} \]

- Lengths shrink in the direction of motion
  - \( d_T \) = rest length, \( d_L \) = observed length
  - At \( v_{rel} = 1 \), \( d_L = 0 \) !!!
Scissors Paradox (Problem 3-14a)

A long straight rod, inclined relative to the x-axis, moves downward at a uniform speed (see above diagram).

- What is the speed of the intersection point A of the rod and the x-axis?
  - \( y = x \tan \theta \Rightarrow dy = dx \tan \theta \Rightarrow \frac{dy}{dt} = \frac{dx}{dt} \tan \theta \Rightarrow v_A = \frac{v_{rod}}{\tan \theta} \)

- Point A can move faster than the speed of light.
  - We have \( v_A > 1 \) when \( v_{rod} > \tan \theta \) (can make \( \theta \) small).
  - You can't transmit information along this point – no signal (matter or energy) is moving with it.
  - A different part of the scissor half at each point (different atoms!)

Searchlight (Problem 3-14c)

- A search light sweeps from A to B
  - Points A and B are located on a plane and are the same distance from the search light.
  - How far away must the searchlight be so that the beam will travel faster from A to B than a light signal could travel from A to B?
    - Derivation at left
  - Suppose A fires a laser at B when searchlight goes by. B ducks when the searchlight passes him.
    - Has the warning message traveled faster than light?
    - No, no matter or energy is transferred from A to B.
    - Different photons arrive at each point.
  - This is just a timing problem.
    - B doesn't really know if A fired!

We have \( x = r \tan \theta \) for \( r >> x \)

\[
\frac{dx}{dt} = r \frac{d\theta}{dt}
\]

Or with \( v_b = \) speed along plane

\[
\frac{d\theta}{dt} = \frac{v_b}{r}
\]

Exceeds speed of light when \( \frac{d\theta}{dt} > \frac{c}{r} \)
An observer in New York sees a rocket with velocity, $v$, is traveling eastward from the West. What is its apparent speed?
- As the rocket passes the Golden Gate Bridge (GGB) a bright flash is emitted.
- When the rocket passes the Gateway Arch (GA) in St. Louis another flash is emitted.
- Thus the images carried by the flashes are separate by a distance $v\Delta t$.
- Now the elapsed time between the images is $(1-v)\Delta t$.
- Thus the apparent (approaching) speed seen in NYC is:
  \[ v_a = \frac{v\Delta t}{(1-v)\Delta t} = \frac{v}{1-v} \implies v = \frac{v_a}{1+v_a} \]
- For $v_a = 4$, $v = 4/5$ and for $v_a = 99$, $v = 0.99$.

Consider an observer in San Francisco watching the rocket move away from them towards the east with velocity, $v$. What apparent speed does she see?
- The second flash occurs at the Gateway Arch (GA).
- A third flash is emitted a time $\Delta t$ later and proceeds toward the observer.
- Thus the images carried by the flashes are separate by a distance $v\Delta t$.
- Now the elapsed time between the images is $(1+v)\Delta t$.
- Thus the apparent (receding) speed seen in SFO is:
  \[ v_r = \frac{v\Delta t}{(1+v)\Delta t} = \frac{v}{1+v} \implies v = \frac{v_r}{1-v_r} \]
- For $v = 4/5$, $v_r = 4/9$ and for $v = 0.99$, $v_r = 0.497$. 
Faster than light travel disallowed

- We will use the Lorentz transformation to illustrate why no material object can travel faster than the speed of light.
  - This is from Box L-1 in your textbook (page 108)
- Here is the scenario:
  - A peace treaty is signed (event A)
  - A Federation starship leaves the signing at 0.6 light speed
  - Four years later the Klingons break the treaty and launch a faster than light missile (event B)
  - The missile travels at three times light speed
  - After passing a couple of federation colonies, the missile destroys the starship (event C)
    - The colonies can’t warn the starship because the missile travels faster than their light signal
- To help visualize the problem we will introduce space time (S-T) diagrams

FTL problem, pt. 1 – Klingon frame

- Klingon world view:
  - Treaty signed on Klingon home world 4 years before launch of FTL missile.
- In Klingon “lab” frame
  - \( x_A = 0 \) yr, \( t_A = -4 \) yr
  - \( x_B = 0 \) yr, \( t_B = 0 \) yr
  - \( x_C = 3 \) yr, \( t_C = 1 \) yr
- Starship velocity
  - \( v_{rel} = 0.6 \) light speed
  - \( \Rightarrow \gamma = 1.25 \)
- We will look at coordinates of the events in the starship frame
  - Use inverse Lorentz transformation

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v_{rel}^2}{c^2}}} = \frac{1}{\sqrt{1 - 0.6^2}} = 1.25 \]
FTL problem, pt. 2 - Transforming

- Transforming to starship frame
- Event A (Treaty Signing)
  \[ t'_A = -\gamma v_{rel} x_A + \gamma t_A \]
  \[ = -0.6 \times 1.25 \times 0 \text{yr} + 1.25 \times (-4 \text{yr}) \]
  \[ = -5 \text{yr} \]
  \[ x'_A = \gamma x_A - \gamma v_{rel} t_A \]
  \[ = 1.25 \times 0 \text{yr} - 0.6 \times 1.25 \times (-4 \text{yr}) \]
  \[ = 3 \text{yr} \]
- Event C (Missile Impact)
  \[ t'_C = -\gamma v_{rel} x_C + \gamma t_C \]
  \[ = -0.6 \times 1.25 \times 3 \text{yr} + 1.25 \times 1 \text{yr} \]
  \[ = -1 \text{yr} \]
  \[ x'_C = \gamma x_C - \gamma v_{rel} t_C \]
  \[ = 1.25 \times 3 \text{yr} - 0.6 \times 1.25 \times 1 \text{yr} \]
  \[ = 3 \text{yr} \]
- Now plot starship S-T diagram

FTL problem, pt. 3 – Starship frame

- Starship view:
  - S-T diagram at left
- In starship frame
  - \( x'_A = 3 \text{yr}, t'_A = -5 \text{yr} \)
  - \( x'_B = 0 \text{yr}, t'_B = 0 \text{yr} \)
  - \( x'_C = 3 \text{yr}, t'_C = -1 \text{yr} \)
- What's going on?
  - The impact occurs before the launch!
  - This can't be – our missile is going backwards in time
- Can't happen
  - Cause and effect break down for FLT objects
  - The S-T interval between B and C is spacelike (recall lecture 5, slide 11)